θ-SPACES AND PERFECT IRREDUCIBLE MAPPINGS OF TOPOLOGICAL SPACES

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Abstract

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θ-SPACES AND PERFECT IRREDUCIBLE MAPPINGS OF TOPOLOGICAL SPACES

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In the present paper the notion of a θ-space is introduced, generalizing the notion of a proximity space compatible with a completely regular space.

At the same time, θ-spaces exist on every regular space, whereas proximity spaces are compatible only with completely regular spaces. A connection is given between θ-spaces on a regular space X and bicompact extensions of all its perfect irreducible preimages. In this connection, a theorem of Yu. M. Smirnov ([4]) on the one-to-one correspondence between proximity spaces compatible with a given completely regular space X and bicompact extensions of the space X is generalized. The paper also introduces the notion of a θ-mapping of θ-spaces and studies its connection with mappings of bicompact extensions of perfect irreducible preimages of topological spaces.

Let X be a regular space. We shall say that a θ-proximity is given on X if for any two subsets $A \subset X$ and $B \subset X$, either $A\theta B$ or $A\bar{\theta}B$ is specified and the following axioms are satisfied:

I. $A\theta B \Rightarrow B\theta A$.

II. $A\bar{\theta}B_i,\ i = 1, 2 \Leftrightarrow A\bar{\theta}(B_1 \cup B_2)$.

III. $\emptyset\bar{\theta}X$.

IV. $\{x\}\theta A \Rightarrow x \in [A]$.

V. $A\bar{\theta}B \Rightarrow$ there exists such a $C = \langle[C]\rangle \supset A$ that $A\bar{\theta}(X \setminus [C])$ and $C\bar{\theta}B$ *.

Axioms I–IV coincide with the corresponding axioms of a proximity space compatible with the given topological space, and it is easy to see that axiom V is a weakening of the normality axiom of a proximity space compatible with a topological space.

An example of a θ-proximity on a regular space X is the following relation: $A\bar{\theta}_a B \Leftrightarrow$ there exist disjoint neighborhoods of the sets A and B. It is easy to verify that all the axioms are satisfied. It will be shown below that the θ-proximity θ_a is maximal.

Theorem 1. *Let $f : Z \to X$ be a perfect irreducible mapping of a completely regular space Z onto a space X, and let bZ be a bicompact extension of the space Z. Then the bicompactum bZ generates on X the following θ-proximity:*

$$A\bar{\theta}B \iff [f^{-1}A]_{bZ} \cap [f^{-1}B]_{bZ} = \emptyset.$$

We shall call a regular space X with a θ-proximity given on it a θ-space.

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* By $\langle D \rangle$ is denoted the interior of the set D.

The main result of the paper is

Theorem 2. *Every θ-space on a regular space X determines a completely regular space X_θ, a perfect irreducible projection $\pi_{X\theta} : X_\theta \to X$ of the space X_θ onto X, and a bicompact extension $b_\theta X_\theta$, which generates the given θ-proximity.*

Remark. The space $b_\theta X_\theta$ is the space of maximal centered systems $\tau = \{H\}$ of open subsets H of the space X with the following θ-regularity condition: for every $H \in \tau$ there exists an $H' \in \tau$ such that $H' \, \theta \, (X \setminus [H])$*.

Theorem 3. *Let $f_1 : Z_1 \to X$ and $f_2 : Z_2 \to X$ be perfect irreducible mappings of completely regular spaces Z_1 and Z_2 onto X. If the bicompact extensions $b_1 Z_1$ and $b_2 Z_2$ generate on X the same θ-space, then there exists a homeomorphism $g : b_1 Z_1 \to b_2 Z_2$ such that $gZ_1 = Z_2$ and $f_1 = f_2 g$.*

Corollary 1. *There exists a one-to-one correspondence between the partially ordered set of all θ-spaces on a regular space X* ***and the set of all pairs*** $(bZ, f : Z \to X)$***, where*** bZ ***is a bicompact extension of the space*** Z ***and*** f ***is a perfect irreducible mapping onto*** X.*

Corollary 2. *There exists a maximal θ-space on a regular space X.*

Corollary 3. *For any two perfect irreducible mappings $f_1 : Z_1 \to X$ and $f_2 : Z_2 \to X$ onto X there exists a perfect irreducible mapping $f : Z \to X$ onto X and perfect irreducible mappings $g_1 : Z \to Z_1$ onto Z_1 and $g_2 : Z \to Z_2$ onto Z_2 such that $f_1 g_1 = f = f_2 g_2$, and moreover the mapping f is minimal in the sense that mappings f', g_1' and g_2' with the same relations factor through the mappings f, g_1 and g_2, respectively***.*

S. Iliadis proved [2] that the space B of maximal centered systems of open sets of a regular space X is the Čech extension of the absolute of the space X. A maximal centered system of open sets in a regular space X obviously satisfies the θ-regularity condition for the θ-proximity θ_a. Hence, by the remark to Theorem 2 and Corollary 1, θ_a is the maximal θ-proximity.

Definition. A perfect irreducible mapping $f : X \to Y$ of a θ-space X onto a θ-space Y will be called a *θ-mapping* if $A, B \subset Y$ and

$$\overline{A}\,\theta\,B \Rightarrow \langle f^{-1}[A]\rangle\,\overline{\theta}\,\langle f^{-1}[B]\rangle.$$

Theorem 4. *Every θ-mapping $f : X \to Y$ generates a perfect irreducible mapping $f_\theta : b_\theta X_\theta \to b_\theta Y_\theta$ such that $f_\theta X_\theta = Y_\theta$ and $f\pi_{X\theta} = \pi_{Y\theta} f_\theta$.*

In accordance with (3), we shall call a multivalued mapping $f : X \to Y$ of θ-spaces a *multivalued θ-mapping* if there exists a θ-space Z and θ-mappings $f_X : Z \to X$, $f_Y : Z \to Y$ such that $f = f_Y f_X^{-1}$.

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* The notion of a centered system of open sets with a certain regularity condition was introduced by P. S. Aleksandrov [1] in the construction of the maximal bicompact extension.

** The order in this set is analogous to the order in the set of proximity spaces compatible with the given space.

*** Two pairs $(bZ, f : Z \to X)$ and $(b'Z', f'Z' \to X)$ are identified if there exists a homeomorphism $g : bZ \to b'Z'$ such that $gZ = Z'$ and $f = f'g$. $(bZ, f) \geq (b'Z', f') \iff$ there exists a perfect irreducible mapping $g : bZ \to b'Z'$ such that $gZ = Z'$ and $f = f'g$. It is easy to verify that pairs (bZ, f) and $(b'Z', f')$ for which both inequalities $(bZ, f) \leq (b'Z', f')$ and $(bZ, f) \geq (b'Z', f')$ hold are identified.

**** A mapping $f' : Z' \to X$ factors through a mapping $f : Z \to X$ if there exists a mapping $g : Z' \to Z$ such that $f' = fg$.

Theorem 5. Every multivalued θ-mapping $f :\ X \xleftarrow{f_X} Z \xrightarrow{f_Y} Y$ generates a multivalued perfect irreducible mapping

$$f_\theta :\ b_\theta X_\theta \overset{\varphi}{\leftarrow} b_\theta Z_\theta \overset{\psi}{\rightarrow} b_\theta Y_\theta,$$

such that $\varphi^{-1} X_\theta = \psi^{-1} Y_\theta$ and $f\pi_{X\theta} = \pi_{Y\theta} f_\theta$.

Theorem 6. Let θ-spaces X and Y be given. Then for every multivalued perfect irreducible mapping $f_\theta :\ b_\theta X_\theta \overset{\varphi}{\leftarrow} B \overset{\psi}{\rightarrow} b_\theta Y_\theta$ satisfying $\varphi^{-1} X_\theta = \psi^{-1} Y_\theta$, there exists a multivalued θ-mapping $f :\ X \xleftarrow{f_X} Z \xrightarrow{f_Y} Y$ such that

$$B = b_\theta Z_\theta, \qquad f\pi_{X\theta} = \pi_{Y\theta} f_\theta, \tag{*}$$

and, among all multivalued θ-mappings satisfying condition $(*)$, there exists a maximal one in the factorization sense.

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Note: Figure translations are in progress. See original paper for figures.