ON THE CONTINUITY OF MAPPINGS AND TRANSLATIONS

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Abstract

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MATHEMATICS

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ON THE CONTINUITY OF MAPPINGS AND TRANSLATIONS

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1. Let a topological space R be given, in which a complete system of neighborhoods is defined (1). We shall say that a complete system of neighborhoods is defined for every set $M \subset R$, if for M a system $\Sigma(M)$ of sets open in R is given, possessing the following properties: a) for every set $U \in \Sigma(M)$ the inclusion $M \subset U$ holds; b) for every pair of sets U and V from $\Sigma(M)$ there exists a set $W \in \Sigma(M)$ such that $W \subset U \cap V$; c) for every set $U \in \Sigma(M)$ and every set $M' \subset \overline{M}$ there exists a set $V \in \Sigma(M')$ such that $V \subset U$; d) if $M' \subset M$, then for every set $U \in \Sigma(M)$ there is a set $U' \in \Sigma(M')$ such that $U' \subset U$; e) if a sequence of points convergent in R is almost entirely contained in an arbitrary set U of the system $\Sigma(M)$, then its limit is a point of contact of the set M; f) if the set M consists of one point, then $\Sigma(M)$ coincides with the complete system of neighborhoods of this point. The totality of all neighborhoods of all sets in R will be called the complete system of generalized neighborhoods of the space R. It is assumed that all topological spaces occurring in the article are, once and for all, equipped with neighborhoods and generalized neighborhoods; moreover, in metric spaces, as neighborhoods there are taken, as usual, all possible spheres of points, and as generalized neighborhoods—all possible "spheres" of sets.

2. A point sequence $x_k \in R,\ k = 1, 2, \dots$, will be called bounded in R if there exists a neighborhood of the space R containing almost all elements of this sequence. We shall assume that sequences not bounded in R can be distinguished from one another by the character of unboundedness or by order of growth.

We shall distinguish point sequences in R by types. Every point sequence convergent in R will be assigned to the zeroth type, a bounded one—to the second type, an unbounded sequence with growth not exceeding a certain order—to the fourth type, and, finally, an arbitrary point sequence in R—to the sixth type.

3. Let a sequence of sets $M_k,\ k = 1, 2, \ldots$, be given in R. A sequence of sets $M_k^*,\ k = 1, 2, \dots$, will be called a daughter sequence for the sequence M_k,

if for every k the inclusion $M_k^* \subset M_k$ is satisfied.

We shall say that a sequence of sets $M_k,\ k = 1, 2, \ldots$, converges in R of the i-th type to the set $M \subset R$ $(i = 0, 2, 4, 6)$, if for every neighborhood V of the set M and every sequence of points $x_k,\ k = 1, 2, \ldots$, of the i-th type and daughter for the sequence M_k, under the condition that the set of such sequences is nonempty, there is a number k_0 such that from the inequality $k \geq k_0$ there follows the inclusion $x_k \in V$.

We shall say that a sequence of sets $M_k,\ k = 1, 2, \ldots$, converges in R of the $(i+1)$-st type to the set $M \subset R$ $(i = 0, 2, 4, 6)$, if it converges in R to the set M of the i-th type and for every point $x \in M$

there is a sequence of points $x_k,\ k = 1, 2, \ldots$, subordinate to the sequence M_k, for which the point x is a limit point.

If a sequence of sets $M_k,\ k = 1, 2, \ldots$, converges in R of the seventh type to a set M, then it converges in R of the seventh type also to the set $\overline{M}$, the latter being uniquely determined and naturally called the **limit of the seventh type** of the sequence M_k.

Let σ_i denote the type of convergence in R of a sequence of sets to some set $(i = 0, 1, 2, 3, 4, 5, 6, 7)$. It is easy to see that the following implications hold:

$$\sigma_6 \to \sigma_0, \qquad \sigma_6 \to \sigma_2, \qquad \sigma_6 \to \sigma_4;$$

$$\sigma_7 \to \sigma_1, \qquad \sigma_7 \to \sigma_3, \qquad \sigma_7 \to \sigma_5;$$

$$\sigma_1 \to \sigma_0, \qquad \sigma_3 \to \sigma_2, \qquad \sigma_5 \to \sigma_4, \qquad \sigma_7 \to \sigma_6.$$

4. Let two spaces* R_1 and R_2 be given. We shall say that a translation B from R_1 into R_2 is defined if a rule is specified by which a set $P_B \subset R_1$ is determined, and to each element x of P_B there is assigned a definite set η of elements of R_2. In this case we shall write $\eta = Bx$. As x ranges over P_B, $\eta \subset R_2$. The set P_B will be called the **domain of definition** of the translation B. The union of all sets $\eta = Bx$, as x ranges over P_B, will be denoted by Q_B and called the **range** of the translation B. It is clear that B is a translation of P_B onto Q_B: $\eta = Bx$, x ranges over P_B, and Q_B is the union of the sets $\eta = Bx$, x ranges over P_B.

If the set $\eta = Bx$ consists of one point, then x is called a **point of single-valuedness** of the translation B. If all points of P_B are points of single-valuedness for B, then B is a mapping of P_B onto Q_B.

If a translation B of the space P_B onto Q_B is defined, then thereby also defined is the translation which assigns to each point $y \in Q_B$ the set ξ of all elements x of P_B for which $y \in Bx$. This translation is called the **inverse** of B and is denoted by the symbol B^{-1}. If the inverse of the translation B is a mapping, then the translation B is called **one-to-one**; otherwise, **many-to-one**.

Assign to each point $x \in P_B$ a definite subset B^*x of the set Bx. As a result there arises a translation B^* of the space P_B into Q_B, which we shall call **subordinate** to the translation B.

We shall say that a family of mappings B_α **exhausts the translation** B if, for every α, the mapping B_α is subordinate to B, and for every $x \in P_B$ the equality

$$Bx = \bigcup_\alpha B_\alpha x$$

holds.

5. Let R_1 and R_2 be topological spaces. A translation B from R_1 into R_2 will be called **complete** if, for every $x \in P_B$, the set Bx is closed. A translation B will be called **regular** if, for every $x \in P_B$, the set Bx is bounded. A translation B from R_1 into R_2 will be called **continuous at the point** $x = x_0$ **of the** i**-th type** ($x_0 \in P_B,\ i = 0,1,2,3,4,5,6,7$) if, for every sequence of points $x_k \in P_B,\ k = 1,2,\dots,$ converging in R_1 to the point x_0, the sequence $Bx_k,\ k = 1,2,\dots,$ converges in R_2 of the i-th type to Bx_0. A translation B from R_1 into R_2 will be called **continuous of the** i**-th type** if at each point of the set P_B it is continuous of the i-th type.

Theorem 1. *If a translation B from R_1 into R_2 is continuous of the i-th type* $(i = 0,1,2,3,4,5,6,7)$*, then every restriction of the translation B is continuous of the i-th type.*

Theorem 2. *If a mapping B of the space R_1 into R_2 is continuous and, for every sequence of points $y_k \in Q_B,\ k = 1,2,\dots,$ converging in R_2 to a point $y_0 \in Q_B$, for the sequence of sets $B^{-1}y_k,\ k = 1,2,\dots,$*

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* In this section, by a space is meant a nonempty fixed set of some elements (?).

there exists a daughter sequence of points of zero type, then the translation B^{-1} is discontinuous of zero type at the point y_0.

Theorem 3. If the mapping B of the space R_1 into R_2 is discontinuous and open, then the translation B^{-1} is discontinuous of the first type.

Theorem 4. If the mapping B of the space R_1 into R_2 is open, then the image of every closed set is a closed set.

6. Let R be a metric space with metric ρ. Consider the functional

$$\Psi_x y = \rho(x, y), \quad y \text{ ranges over } \overline{Q},$$

where Q is a nonempty subset of R and $x \in R$.

Theorem 5. If the set Q is dense in R, or if every sphere of the space R is compact, then the functional Ψ_x has a minimum; moreover, the set M_x of elements minimizing this functional is a full regular translation of the space R into $\overline{Q}$, discontinuous of the sixth type.

7. Let R_1 and R_2 be metric spaces with metrics ρ_1 and ρ_2, respectively. A family of mappings B_α from R_1 into R_2, defined on one and the same set $P \subset R_1$, will be called **uniformly discontinuous** at a point $x_0 \subset P$ if, for every $\varepsilon > 0$, there exists a neighborhood $V_{x_0} \subset R_1$ of the point x_0 such that the inclusion

$$B_\alpha(V_{x_0} \cap P) \subset s(B_\alpha x_0, \varepsilon) \cap B_\alpha P.$$

holds.

Theorem 6. Let a translation B from R_1 into R_2 be given, and let a family of mappings B_α exhausting it be defined. Then, if the family B_α is uniformly discontinuous at each point of the set P_B, the translation B is discontinuous of the seventh type.

Denote by R_3 the topological product of the spaces R_1 and R_2, with metric $\rho_3 = \max(\rho_1, \rho_2)$. If K_1 and K_2 are compact-in-themselves subsets of the spaces R_1 and R_2, respectively, then K_3, equal to the topological product of K_1 by K_2, will be a compact-in-itself subset of the space R_3, and conversely.

Theorem 7. Let a real continuous functional $F(x, y)$ be given on a compact-in-itself set K_3 of the space R_3, where x ranges over K_1 and y ranges over K_2. We shall regard $F(x, y)$ as a functional only of y for fixed x: $F_x y = F(x, y)$, y ranges over K_2. Then the set N_x of elements minimizing the functional $F_x y$, where y ranges over K_2, is a full, regular translation of K_1 into K_2, discontinuous of the sixth type.

8. **Theorem 8.** Let a translation B be given from the topological space R_1 onto the topological space R_3. Suppose, moreover, that a translation T of the subspace P_B onto the space R_2 and a translation S of the space R_2 onto the space R_3 are given such that $B = ST$.

A. Then, if the space R_2 is topological, the translation S is discontinuous of the zero (first) type, and the translation T is full and discontinuous of the zero (first) type, then the translation B is discontinuous of the zero (first) type.

B. Then, if the space R_2 is metric and each of its closed spheres is compact, the translation T is full and regular, and the translations S and T are discontinuous of the sixth type, then the translation B is discontinuous of the sixth type.

Remark. Theorem 8B is also true in the case when the set Q_B is a proper part of the space R_3.

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CITED LITERATURE

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2. L. Collatz, *Funktionalanalysis und numerische Matematik*, Berlin, 1964.

Note: Figure translations are in progress. See original paper for figures.