Reports of the Academy of Sciences of the USSR

V. G. Khryshtun

1967

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1967, Volume 177, No. 3

UDC 517.916:517.94:517.944(045) **MATHEMATICS**

V. G. Khryshtun

CLASSES OF FUNCTIONS QUASIANALYTIC WITH RESPECT TO AN ORDINARY LINEAR DIFFERENTIAL OPERATOR, AND THEIR APPLICATION

(Presented by Academician S. L. Sobolev, February 4, 1967)

1. Let the operator L be given by the formula

$$L = D_x^n + \sum_{k=0}^{n-1} p_k(x) D_x^k \left(D_x = \frac{d}{dx},\ D_x^0 = I \right), \tag{1}$$

where the coefficients $p_k(x)$ $(k = 0, 1, \dots, n-1)$ are continuous functions on an arbitrary fixed interval Δ of the real axis, and I is the identity operator. By M we denote another operator

$$M = D_y^{n_1} + \sum_{k=0}^{n_1-1} q_k(y) D_y^k \left(D_y = \frac{d}{dy} \right), \tag{2}$$

of the same type, with coefficients $q_k(y)$ continuous on an arbitrary fixed interval Δ_1 of the real axis.

Denote by $\{k_i\}_{i=0}^{\infty}$ and $\{\nu_j\}_{j=0}^{\infty}$ arbitrary fixed monotonically increasing sequences of positive integers, and by $\{M_{k_i}\}_{i=0}^{\infty}$ and $\{N_{\nu_j}\}_{j=0}^{\infty}$ arbitrary fixed sequences of positive numbers.

An infinitely L-differentiable (see ([5])) function $f(x)$ on the interval Δ will be said to belong to the class $C^L\{M_{k_i}\}$ if, for every closed interval $\delta \subset \Delta$, there exist constants N and C (depending on L, f, and δ) such that

$$\left| D_x^{r_i} L^{q_i} f(x) \right|_\delta \le N C^{k_i} M_{k_i}, \qquad k_i = n q_i + r_i, \quad 0 \le r_i \le n-1, \quad q_i \ge 0, \tag{3}$$

where q_i and r_i are integers.

The class $C^L\{M_{k_i}\}$ of functions $f(x)$ defined on the interval Δ will be called a quasi-L-analytic class if, from the equalities $D_x^{rL^{qf}}(a) = 0$, holding at some point $a \in \Delta$ for all $r = 0, 1, \dots, n-1$ and $q = 0, 1, 2, \dots$, it follows that $f(x) \equiv 0$ on Δ.

A function $g(x, y)$, defined in the domain $x \in \Delta$, $y \in \Delta_1$, infinitely L-differentiable with respect to x and infinitely M-differentiable with respect to y, will be said to belong to the class

$$C^{L,M}\{M_{k_i}; N_{\nu_j}\},$$

if for every closed interval $\delta \subset \Delta$ and every closed interval $\delta_1 \subset \Delta_1$ there exist constants A and B (depending on L, M, g, δ, and δ_1) such that

$$\left| D_x^{r_i} D_y^{\sigma_j} L^{q_i} M^{\mu_j} g(x, y) \right|_{\delta \times \delta_1} \le A B^{k_i + \nu_j} M_{k_i} N_{\nu_j}, \tag{4}$$

$$k_i = n q_i + r_i, \quad 0 \le r_i \le n-1, \quad q_i \ge 0;$$

$$\nu_j = n_1 \mu_j + \sigma_j, \quad 0 \le \sigma_j \le n_1 - 1, \quad \mu_j \ge 0,$$

where $q_i, r_i, \mu_j, \sigma_j$ are integers.

Suppose that the function $\sigma(x, y)$ in the domain $x \in \Delta$, $y \in \Delta_1$ is the sum of the series

$$\sigma(x, y) = \sum_{p=0}^{\infty} b_p(x, y)(x - x_1)^p \qquad (x_1 \in \Delta) \tag{5}$$

and the functions $b_p(x, y)$ are continuous in the domain $\Delta \times \Delta_1$. Then the function

$$\varphi(x) = \sum_{p=0}^{\infty} a_p (x - x_1)^p, \tag{6}$$

whose coefficients a_p are given by the formulas

$$a_p = \max_{\delta \times \delta_1} |b_p(x, y)|, \qquad p = 0, 1, 2, \dots$$

will be called an exact Cauchy majorant of the function (5) in the closed domain $\delta \times \delta_1 \subset \Delta \times \Delta_1$. If for every closed domain $\delta \times \delta_1 \subset \Delta \times \Delta_1$ the function (6) is an entire function of order of growth ρ, then this number ρ will be called the order of conditional growth with respect to the variable x of the function (5) in the domain $\Delta \times \Delta_1$.

We shall assume that

$$\lim_{i\to\infty} \sqrt[k_i]{M_{k_i}} = \infty,$$

and then, for the sequence of points $(k_i, \ln M_{k_i})$, one can construct the Newton polygon (see (2)). The sequence $\{M^c_{k_i}\}_{i=0}^{\infty}$ will be called the convex regularization by means of the logarithms of the sequence $\{M_{k_i}\}_{i=0}^{\infty}$, if the number $\ln M^c_{k_i}$ is equal to the ordinate of the Newton polygon at the point k_i.

If $L = D_x$ and $k_i = i$, then the classes of functions $C^L\{M_i\}$ coincide with the classes of functions $C\{M_i\}$, whose properties are set forth in ([2]).

In the present note we establish that the conditions

$$\lim_{i\to\infty} \sqrt[k_i]{M_{k_i}} = \infty, \qquad \sum_{i=1}^{\infty}(k_i - k_{i-1})(M_{k_{i-1}}/M^c_{k_i})^{1/(k_i-k_{i-1})} = \infty \tag{7}$$

are sufficient for the quasi-L-analyticity of the classes of functions $C^L\{M_{k_i}\}$, if the coefficients of the operator L of the form (1) are continuous. Under additional restrictions on the coefficients $p_k(x)$ we single out those operators L for which it has been possible to establish that the conditions (7) are necessary for the quasi-L-analyticity of the classes of functions $C^L\{M_{k_i}\}$. We use these results in investigating the properties of solutions of the equation

$$L^s f(x,y) = M^m f(x,y), \tag{8}$$

satisfying the initial data

$$D_x^r L^q f(x,y)\big|_{x=x_1} = \varphi_{nq+r}(y), \qquad r = 0,1,\dots,n-1;\ q = 0,1,\dots,s-1, \tag{9}$$

where s and m are natural numbers. The solution of problem (8)–(9) is sought in the classes of functions $C^{L,M}\{M_{k_i}; N_{v_j}\}$, the initial data (9) being chosen in such a way that these classes are quasi-analytic with respect to one of the operators L or M. The direction of the investigation was chosen under the influence of A. N. Tikhonov' s work ([3]) and M. K. Fage' s work ([6]).

We also note that the sufficient conditions for the quasi-L-analyticity of the note (7) are special cases of the conditions (7), and that A. A. Tyanovskii ([4]) also arrived at the conditions (7) as sufficient conditions for quasi-L-analyticity.

2. **Theorem 1.** *Conditions (7) are sufficient for the quasi-L-analyticity of the classes of functions $C^L\{M_{ki}\}$, if L is an operator (1) of arbitrary order $n \geq 1$ with continuous coefficients.*

The proof is carried out in the same way as in note (7), with the aid of the generalized Bang formula.

Theorem 2. *Conditions (7) are necessary for the quasi-L-analyticity of the classes of functions $C^L\{M_k\}$ for the following operators L of the form (1):*

1) L is a first-order operator with coefficient $p_0(x)$ continuous on the interval Δ;
2) L is a second-order operator with coefficients continuous on the interval Δ, and the coefficient $p_1(x)$ has a second derivative, while $p_0(x)$ has a first derivative to the right (or to the left) of some point $x_1 \in \Delta$;
3) L is an operator of arbitrary order, all coefficients of which are continuous on the interval Δ and are analytic functions in an arbitrarily small neighborhood of one of the endpoints of the interval Δ;
4) L is an operator of arbitrary order, all coefficients of which are continuous on the interval Δ and, in an arbitrarily small neighborhood of some point $x_1 \in \Delta$, are constant.

For the first three types of operators L, the proof is carried out by means of a transformation operator (see (1)).

In the next four theorems we establish some properties of solutions of problem (8)–(9).

Theorem 3. *Let $n_1 m \leq ns$, and let on the interval Δ_1 the functions (9) belong to the class $C^M\{N_j\}$, where $N_j = j!$. Then in a neighborhood of each point $(x_1, y_1) \in \Delta \times \Delta_1$ there exists a unique solution of problem (8)–(9), belonging to the class $C^{L,M}\{M_{nsi}; N_j\}$, where $M_{nsi} = (nsi)!$.*

If $n_1 m < ns$, then the order of conditional growth with respect to the variable x of the solution of problem (8)–(9) does not exceed the number $ns/(ns - n_1 m)$.

Theorem 4. *Let $n_1 m < ns$, let ε be an arbitrary positive number, and let β and γ be defined by the equalities*

$$ns/(ns - \beta) = ns/(ns - n_1 m) + \varepsilon, \qquad \gamma = \beta / n_1 m.$$

If on the interval Δ_1 the functions (9) belong to the class $C^M\{N_j\}$, where $N_j = (\gamma j)^{\gamma j}$, then in the domain $\Delta \times \Delta_1$ there exists a unique solution of problem (8)–(9), belonging to the class $C^{L,M}\{M_{nsi}; N_j\}$, where $M_{nsi} = (nsi)!$. The order of conditional growth with respect to the variable x of the solution of problem (8)–(9) is not greater than the number $ns/(ns - n_1 m) + \varepsilon$.

Theorem 5. *Let $n_1m > ns$ and $\alpha = ns/n_1m$. If on the interval Δ_1 the functions (9) belong to the class $C^M\{N_j\}$, where $N_j \leq \Gamma(\alpha j + 1)$, then in a neighborhood of each point $(x_1, y_1) \in \Delta \times \Delta_1$ there exists a unique solution of problem (8)–(9), belonging to the class $C^{L,M}\{M_{nsi}; N_j\}$, where $M_{nsi} = (nsi)!$ and $N_j \leq \Gamma(\alpha j + 1)$. The order of conditional growth with respect to the variable y of the solution of problem (8)–(9) is not greater than $n_1m/(n_1m - ns)$;*

Theorem 6. *If the operator L satisfies the conditions of Theorem 2, $n_1m > ns$, and the order of conditional growth with respect to the variable y of the solution of problem (8)–(9) is greater than $n_1m/(n_1m - ns)$, then the solution is not unique.*

If in Theorem 6 we put $s = m = 1$, $L = D_x$, and $M = D_y^2$, then we obtain the well-known result of A. N. Tikhonov (3).

Computing Center
of the Siberian Branch of the Academy of Sciences of the USSR

Received
27 I 1967

CITED LITERATURE

1. B. M. Levitan, *Operators of generalized shift and some of their applications*, Moscow, 1962.
2. S. Mandelbrojt, *Attached series. Regularization of sequences*, IL, 1955.
3. A. N. Tikhonov, Mat. sborn., 42, No. 2, 199 (1935).
4. A. A. Tynovskii, *Operator Taylor–Bang formulas and some of their applications*, Candidate' s dissertation, Kiev, 1966.
5. M. K. Fage, Tr. Moscow Math. Soc., 7, 227 (1958).
6. M. K. Fage, *Operator-analytic functions of one independent variable*, Lviv, 1959.
7. V. G. Khryshtun, Siberian Math. Journal, 6, No. 6, 1395 (1965).

Note: Figure translations are in progress. See original paper for figures.