

Conditions for the existence of circular motions of Froude' s connected pendulums

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Abstract

The article considers a system of fourth-order differential equations describing, in particular, the motion of coupled Froude pendulums of equal masses suspended on a shaft rotating at a speed Ω . The paper provides upper and lower bounds for the values of the speed Ω in the case of the existence of circular motions of the pendulums, i.e., motions in which the pendulum velocity is positive starting from a certain point in time. Under various assumptions regarding the parameters and functions included in the system, sufficient conditions for the existence of circular motions of the pendulums are derived, and four theorems on the existence of a periodic solution to the system are formulated. As in the author' s previously published works (see bibliography), the study is conducted using the method of projections onto phase planes by introducing comparison equations. Illustrations: 2. Bibliography: 4.

Full Text

Preamble

This study investigates the dynamics of a coupled system of differential equations, building upon the foundational work established in [?] and [?]. We consider a system of the form:

$$\begin{aligned}\ddot{x} + a\dot{x} + f(x) &= N(Q - x) + kF(y - x) \\ \ddot{y} + a\dot{y} + f(y) &= N(Q - y) - kF(y - x)\end{aligned}\tag{1}$$

Under the assumptions (A), let $f(x)$ be a periodic function such that $f(x_1) = f(x_2) = 0$ for $x_1 - x_2 = 2\pi$. We assume $f'(0) > 0$ and $f(x) > 0$ for $x \in (0, x_1)$. The parameters N, Q , and k are constants. The coupling function $F(z)$ is assumed to be bounded, $|F(z)| \leq 1$, and we specifically consider cases where

$f(x) = \sin x$. This system describes the synchronization of coupled oscillators under external torque $N(Q - x)$, as discussed in [?].

By introducing the variables $u = \dot{x}$ and $v = \dot{y}$, we can rewrite the system (1) as a set of first-order equations. Following the methodology in [?], we define the auxiliary function $\Phi(x)$ to simplify the analysis of the phase trajectories:

$$\Phi(x) = f(x) - NQ + k \tag{4}$$

We assume the existence of a periodic solution such that $\Phi(x + 2\pi) = \Phi(x)$ and $\int_0^{2\pi} \Phi(x)dx < 0$ (6), (7). For the system to exhibit stable equilibrium points, the parameters Q, N , and k must satisfy the condition:

$$0 < Q < \frac{\max f(x) \pm k}{N} \tag{10}$$

As shown in

, the phase portrait of the system is characterized by the behavior of the function $\Phi(x)$. Let η_0 be a root such that $\Phi(\eta_0) = 0$. We define the regions of stability based on the sign of $\Phi(x)$ relative to the equilibrium points η_1 and η_2 , where $\eta_1 - \eta_2 = 2\pi$.

The existence of limit cycles and the stability of the rotational motions are governed by the energy integrals S_1 and S_2 (see

). According to the criteria in [?], the condition for the absence of out-of-phase rotations is given by the inequality:

$$S_1(\eta_0) > S_2(\eta_0) \tag{14}$$

For the trajectories $u = S_1(x)$ in the interval $\eta_0 < x < \eta_1$, we can derive the following upper bound:

$$S_1(x) < (a + N)(\eta_1 - x) + \sqrt{2 \int_x^{\eta_1} \Phi(x)dx} \tag{16}$$

Conversely, for the trajectory $u = S_2(x)$ in the interval $\eta_2 < x < \eta_0$, the lower bound is expressed as:

$$S_2(x) > (a + N)(\eta_2 - x) + \sqrt{2 \int_{\eta_2}^x \Phi(x)dx} \tag{20}$$

Based on these estimates, we can formulate several theorems regarding the global stability of the system.

PEOPLE OF SOVIET SCIENCE

EVGENY ALEKSEEVICH BARBASHIN

(on his 50th birthday)

On January 17, 1968, Academician of the Academy of Sciences of the Byelorussian SSR Evgeny Alekseevich Barbashin turns 50.

E. A. Barbashin was born into the family of an employee in the village of Uyinskoye, Perm Region. At the age of six, he was left a complete orphan and was raised in a children's home from 1927. Upon graduating from secondary school in 1935, E. A. Barbashin entered the Physics and Mathematics Faculty of Ural University and in 1940, postgraduate studies at Moscow State University. The supervisor of postgraduate student E. A. Barbashin was V. V. Nemytskii. Evgeny Alekseevich immediately became an active participant in the famous seminar on qualitative theory, which was supervised by V. V. Stepanov and V. V. Nemytskii. He was the first to develop the theory of dynamical systems without the assumption of uniqueness. After defending his candidate's dissertation in 1943, E. A. Barbashin was appointed associate professor of the Department of Higher Mathematics of the Ural Polytechnic Institute. In 1948, 1948, Evgeny Alekseevich entered doctoral studies at the V. A. Steklov Mathematical Institute of the USSR Academy of Sciences and in 1951 successfully defended his doctoral dissertation on the topic "The Method of Sections in the Theory of Dynamical Systems." In this work, in particular, for the first time, issues of qualitative study of dynamical systems globally were linked with stability problems of motion. This allowed obtaining a number of new results in stability theory according to Lyapunov in particular, proving the existence of a smooth Lyapunov function in the region of attraction of an asymptotically stable singular point.



In 1952, Evgeny Alekseevich returns to Sverdlovsk to the Department of Higher Mathematics of the Ural Polytechnic Institute. Here his great scientific and organizational work begins on creating the now well-known Sverdlovsk school of differential equations.

At the seminar. At the seminar of E. A. Barbashin, a large and strong collective of Sverdlovsk mathematicians was formed, who have performed and are performing interesting research on a wide range of problems from the theory of differential equations, stability theory, theory of oscillations, theory of optimal control and management. This collective, alongside research into fundamental theoretical questions, always

Figure 1: Figure 1

paid great attention to solving important applied problems. The scientific interests of E. A. Barbashin at that time were focused mainly on two large problems. He studied stability of essentially nonlinear systems as a whole and tasks on stability of motions of nonlinear systems according to the first approximation. As a result, important theorems were proven by him, substantiating the method of Lyapunov functions in application to a new class of problems, methods for effectively constructing Lyapunov functions for specific types of nonlinear systems were developed, and general methods for proving stability theorems according, according to the first approximation were developed, working not only in the field of ordinary differential equations, but also in the field of systems of a more general form (integro-differential equations, etc.). Let us note that the Sverdlovsk seminar of E. A. Barbashin and other research collectives of Sverdlovsk mathematicians and mechanics that grew on its basis are in great friendship and work in close contact with many scientific collectives of our country. We are talking here primarily about mathematicians of Leningrad University, the Institute of Problems of Mechanics of the Academy of Sciences of the USSR, Moscow mathematicians, etc.

In the period 1961–1966, E. A. Barbashin was a fellow of the Mathematical Institute of the Academy of Sciences of the USSR (Sverdlovsk Branch), headed the Department of Mathematical analysis of this Institute. Evgenii Alekseevich played one of the leading roles in creating and strengthening this new large mathematical center in the Urals. At this time, problems of automatic control theory occupied a particularly large place in his research. He published a series of papers on program control, in which the connection of tasks on implementation of program motions with the general theory of stability of motion, with the theory of optimal processes and with the theory of approximation of functions was comprehensively covered. Further, new original and effective methods for stabilization of automatic control systems were developed by him, covering, in particular, dynamic systems with non-regular commutative characteristics that are highly important for applications and difficult for mathematical study. Finally, lately, a new concept attempt of the basic concepts of stability theory of motion was developed by E. A. Barbashin, treating this property in connection with the reaction of the system to external disturbances of one or another class. This concept allowed, in particular, to attract the apparatus of generalized functions to the study of stabilizes. A series of interesting results of E. A. Barbashin, obtained in this direction, entered his monograph "Introduction to the Theory of Stability". Publishing House "Nauka", 1967.

Evgenii Alekseevich educated a large number of scientific workers, including over 25 candidates of sciences.

In 1966, Evgenii Alekseevich was elected an academician of the Academy of Sciences of the BSSR and moves to work in Minsk. He is a member of the editorial board of the All-Union journal "Differential equations", heads the laboratory of applied mathematics and mechanics of the Institute of mathematics of the Academy of Sciences of the BSSR and the department of applied mathematics of the Belorussian State University named after V. I. Lenin.

In 1967, for scientific work, Evgenii Alekseevich was awarded the Order of the Red Banner of Labor.

Evgenii Alekseevich Barbashin is in the prime of his life. He and his students, undoubtedly, will carry out many more profound studies on various questions of the theory of differential equations.

N. P. ERUGIN, N. N. KRASOVSKII

Figure 2: Figure 2

Theorem 1. If the parameters satisfy the inequality:

$$a > \frac{N(-x_2)}{\ln(a + N)} + \sqrt{2J} + \dots \quad (21)$$

then the system (1) does not possess limit cycles of the second kind, and all trajectories converge to the equilibrium state (7).

Theorem 2. For the specific case $f(x) = \sin x$, the condition for synchronization (the absence of asynchronous rotations) is satisfied if:

$$a > \frac{\sqrt{2J} - (a + N)\pi}{\pi} + \dots \quad (24)$$

where J represents the integral of the restoring force over a period.

Further analysis of the phase space shows that when $k < NQ$, the coupling term $kF(y - x)$ acts as a restorative force that facilitates synchronization. If the damping coefficient a is sufficiently large, specifically satisfying:

$$\frac{\sqrt{2J} - 4(a + N)\pi - (a + N)x_1}{\sqrt{2J} - 4(a + N)\pi} > 0 \quad (30)$$

then the system (1) is guaranteed to reach a synchronous state regardless of the initial conditions. These results extend the classical stability criteria for pendulum-like systems to coupled oscillators with linear dissipation and constant external torque.

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Figures

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Figure 3: Figure 3

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Figure 4: Figure 4