ON OPTIMAL CONTROL OF SYSTEMS WITH DISTRIBUTED PARAMETERS

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Abstract

Full Text

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1. In the note [1], certain necessary optimality criteria and a uniqueness theorem were formulated for a class of problems (class A) on the optimization of the functional

$$I(\mu) = \int_\Omega F(p, u_\mu(T,p))\, d\Omega \tag{1}$$

(where $\mu(t)$ are controls), under the condition that the right endpoint of the system trajectory is free.

In the present note, which is a continuation of [1], we shall consider a class of optimal problems (class B), when the functional

$$I(\mu) = \int_\Omega \int_0^T F(t, p, u_\mu, Du_\mu, \mu(t))\, dQ, \tag{2}$$

is minimized, where, as in [1], $u_\mu(t,p)$ is a generalized solution of a parabolic equation with a boundary condition of the third kind and with zero initial data. (For more detail on the notation and the formulation of the problem, see [1, 2].) It is assumed that the controls $\mu(t)$ are measurable vector-functions with values in a certain bounded set of the vector space R^s. In addition, we assume that the right endpoint of the trajectory $u_\mu(t,p)$ is constrained by certain restrictions.

2. Let us make a remark. In contrast to the finite-dimensional case, in infinite-dimensional spaces the problem (of type B) of optimally bringing the trajectory $u_\mu(t,p)$ to a fixed attainable point of this space may lose its meaning as a variational problem; namely, it is not difficult to prove that for class B (for dimension $n \geq 2$) the following assertion is true: between the set of all admissible controls $\mu(t) \in L_2(0,T)$ and the set of all attainable points $u_\mu(T,p) \in L_2(\Omega)$ there exists a one-to-one correspondence.

3. Passing to the main problem B, in connection with the remark we shall assume that the set $\overline{G}$ of possible values of the endpoints of the trajectories $u_\mu(t,p)$ is defined as a generalized neighborhood by means of the functional

$$K(t, u(p)) \equiv \int_\Omega \Phi(t, p, u(p))\, d\Omega$$

$$G(t) = \{u(p) \in L_2(\Omega),\ K(t, u(p)) < 0\},$$

where the function $\Phi(t, p, u)$, together with the derivative Φ'_u, is assumed to be measurable in p and continuous in $t \times u$. It is also assumed that the boundary of the domain G is smooth, i.e., that if $K = 0$, then $\Phi'_u(t, p, u(p)) \neq 0$.

The necessary optimality criteria will be formulated for the functional (2) under the condition that the function F does not explicitly depend on Du_μ. The general case is treated analogously. It is assumed, of course, that the functional (2) has a continuous Fréchet derivative.

Let now $\mu_0(t)$, $0 < t < T_0$, $T_0 > 0$, be optimal elements of our problem, and let $\{v_m(p)\}$ be a complete orthonormal system in $L_2(\Omega)$ of generalized eigenfunctions of the operator associated with the stationary equation of the problem; let $\{\lambda_m\}$ be the corresponding set of eigenvalues. Further, let $\mu_{\varepsilon,\tau}(t)$, $0 < t < T_0$, be a control defined by the equalities (see ([3]))

$$\mu_{\varepsilon,\tau}(t) = \begin{cases} \mu_3(t), & \text{if } t \in (\tau - \varepsilon, \tau), \\ \mu, & \text{if } t \notin (\tau - \varepsilon, \tau). \end{cases}$$

Then the following formulas hold:

a) $\Delta I = I(\mu_{\varepsilon,\tau}) - I(\mu_0) = \varepsilon \delta I + o(\varepsilon)$,

b) $\Delta u = u_{\mu_{\varepsilon,\tau}}(T_0, p) - u_{\mu_0}(T_0, p) = \varepsilon \delta u + o(\varepsilon)$,

where δI and δu are the corresponding first variations.

4. Necessary optimality criteria. Consider several possible cases in our problem B.

I. The end of the trajectory lies inside the domain G. It is not difficult to see that in this case a necessary condition for optimality of the control $\mu_0(t)$ is the nonnegativity of the first variation δI: $\delta I(\tau, \mu) \geq 0$ for almost all $\tau \in (0, T_0)$ and all μ.* Hence the following follows.

Theorem 1. Let $u_{\mu_0}(T_0, p) \in G$. If $\mu_0(t)$ is optimal, then (for almost all τ)

$$\inf_{\mu \in \Pi^s} \left\{ \int_\Omega F(\mu)\, d\Omega + \sum_m \int_\Sigma \int_\tau^T v_m F'_m(t)\, g(\mu)\, e^{-\lambda_m (t-\tau)}\, dt\, d\Sigma \right\} =$$

$$= \int_\Omega F(\mu_0(\tau))\, d\Omega + \sum_m \int_\Sigma \int_\tau^T v_m F'_m(t)\, g(\mu_0(\tau))\, e^{-\lambda_m (t-\tau)}\, dt\, d\Sigma,$$

where Π^s is the control domain; $g(\mu)$ is the right-hand side in boundary conditions of the third kind;

$$F'_m(t) = \int_\Omega F'_u v_m(p)\, d\Omega.$$

Consider the simplest example. Let the functional have the form

$$I(\mu) = \int_0^1 [u_\mu - u(x)]^2\, dx + \gamma \int_0^T \mu^2(t)\, dt, \qquad \gamma > 0, \qquad |\mu| \le 1,$$

and the third boundary-value problem is solved (as in ([4])) for the equation $u_t = u''_{xx}$. By virtue of Theorem 1 we obtain, for all $\tau \in (0, T)$:

$$\mu_0^2(T) + \frac{1}{\gamma} \sum_m \mu_0(\tau) F'_m a_m e^{-\lambda_m (T_0 - \tau)} = \min_{|\mu| \le 1} \left\{ \mu^2 + \frac{1}{\gamma} \sum_m \mu F'_m a_m e^{-\lambda_m (T_0 - \tau)} \right\},$$

i.e. in this case the optimal control is always inertial (continuous).

II. More important for applications is the case where the end of the trajectory lies on the boundary of the domain G, i.e.

$$\int_\Omega \Phi(T_0, p, u_{\mu_0}(T_0, p))\, d\Omega = 0.$$

* Obviously, this case can be included in the class of problems with a free right end, since in deriving formulas a) and b) only local variations of $u_{\mu_\varepsilon}(t, p)$ were used.

Let us first consider the special case when $\delta I = 0$, but the time T_0 is not fixed in advance; namely, let

$$I(T) = \int_0^T \int_\Omega F(t, p)\, dQ \tag{3}$$

and let the functional (3) satisfy the condition

$$I(T_1) > I(T_2), \quad \text{if } T_1 > T_2 \tag{4}$$

(this includes the case of the time-optimal problem).

For this case the following criterion holds:

Theorem 2. *If the functional of problem B satisfies conditions (3) and (4), then for almost all* τ

$$\inf_{\mu\in\Pi^s}\left\{\sum_m \Phi_m \int_\Sigma v_m(p)\, g(\mu)\, d\Sigma\, e^{-\lambda_m(T_0-\tau)}\right\} =$$

$$= \sum_m \Phi_m \int_\Sigma v_m(p)\, g(\mu_0(\tau))\, d\Sigma\, e^{-\lambda_m(T_0-\tau)},$$

where

$$\Phi_m = \int_\Omega v_m(p)\, \Phi'_u(T_0, p, u_\mu(T_0,p))\, d\Omega.$$

Now suppose that for some τ and μ, $\delta I < 0$. In this case it is necessary to introduce into consideration the cone of attainability K (see (3)), defined as the collection of vectors $(\delta I, \delta u)$ issuing from the point $(I(\mu_0(t)), u_{\mu_0}(T_0,p))$ of the space $(-\infty < I < +\infty \times L_2(\Omega))$ and constituting the principal linear parts of the increments ΔI and Δu on the varied controls specified by the equalities

$$\mu_{\tau,\varepsilon,\beta_j^k}(t) = \begin{cases} \mu_j^k, & \text{if } t \in \left(\tau_j - \varepsilon\sum_{p=1}^{k}\beta_j^p,\ \tau_j - \varepsilon\sum_{p=1}^{k-1}\beta_j^p\right), \\ \mu_0(t), & \text{if } t \notin \left(\tau_j - \varepsilon\sum_{p=1}^{k}\beta_j^p,\ \tau_j - \varepsilon\sum_{p=1}^{k-1}\beta_j^p\right). \end{cases}$$

All τ_j are assumed to be distinct Lebesgue points, $\beta_j^k \geq 0$, $j = 1, 2, \dots, r$, $k = 1, 2, \dots, l_j$; $\mu_j^k \in \Pi^s$.

We note that the cone K is a convex set in $(-\infty < I < +\infty \times L_2(\Omega))$. It is also easy to establish that it has no common points with the interior of the angle (I, Φ), defined by the inequalities

$$I \leq I[\mu_0(t)],$$

$$\int_\Omega \Phi'_u \bar{u}(p)\, d\Omega \leq \int_\Omega \Phi'_u u_{\mu_0}(T_0,p)\, d\Omega.$$

Since the angle (I, Φ) is a convex body in the space $(R^1 \times L_2(\Omega))$, by the separation theorem ([5]) there exists a constant $C \geq 0$ such that

$$C\,\delta I + \int_\Omega \Phi'_u(T_0, p, u_{\mu_0}(T_0,p))\, \delta u\, d\Omega \geq 0 \tag{5}$$

for arbitrary vectors $(\delta I, \delta u)$ of the cone K. From (5) we obtain the necessary optimality conditions in the case when $\delta I < 0$ for some τ and μ–Theorem 3, analogous to Theorems 1 and 2 with the obvious changes (following from (5)).

Arguing in the same way as in the derivation of (5), we see that there always (i.e., independently of the sign of δI) exist two constants $C \geqslant 0$ and $\sigma \geqslant 0$, $C + \sigma > 0$, such that

$$C\delta I + \sigma \int_\Omega \Phi'_u \delta u \, d\Omega \geqslant 0$$

for any vectors $(\delta I, \delta u)$ of the cone K. Hence there follows a general optimality criterion for $\mu_0(t)$, independent of the sign of δI, analogous to Theorem 3.

5. All the preceding arguments (with obvious modifications) remain valid if the domain G, formed with the aid of the functional K, is replaced by an arbitrary but convex domain $G^* \subset L_2(\Omega)$.

6. Analogous criteria can also be formulated for more general optimal problems, when the control is concentrated not only on the boundary, but also inside the domain of definition of the parameters, Ω.

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References

1. V. I. Plotnikov, DAN, **170**, No. 2 (1966).

2. V. I. Plotnikov, DAN, **165**, No. 1 (1965).

3. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, B. F. Mishchenko, *The Mathematical Theory of Optimal Processes*, Moscow, 1961.

4. Yu. V. Egorov, *Zhurnal vychislitel̆noi matematiki i matematicheskoi fiziki*, **3**, No. 5 (1963).

5. N. Dunford, J. Schwartz, *Linear Operators (General Theory)*, Moscow, 1962.

Note: Figure translations are in progress. See original paper for figures.