SOME REMARKS ON THE DUALITY OF FUNCTORS IN THE CATEGORY OF ABELIAN GROUPS

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Abstract

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SOME REMARKS ON THE DUALITY OF FUNCTORS IN THE CATEGORY OF ABELIAN GROUPS

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The note contains several facts concerning the duality of functors, introduced in ([1,2]), for the case when the base category is the category of abelian groups. Although the duality operator for functors in the categories of finite and finitely generated abelian groups is not a special case of the general construction ([2]), this difference is inessential. We use the usual notation: $\Sigma_A B = A \otimes B$, $\Omega_A B = \mathrm{Hom}(A, B)$; by $\{S \to T\}$ we denote the set of mappings of the functor S into the functor T, endowed with the structure of an object of the category. By a functor we always mean a covariant additive functor*.

§ 1. The functor $T_{A,B}$. Let $\mathcal{C}$ be any D-category ([2]); A, B its objects. Put

$$T_{A,B}X = \mathrm{Hom}(\mathrm{Hom}(X, A), B).$$

This defines a functor (covariant in X). We note its obvious property: for any functor S there is a natural, in A and B, equality

$$\{S \to T_{A,B}\} = \mathrm{Hom}(SA, B).$$

In particular, it follows from this that if in the category $\mathcal{C}$ there exists an object K such that $T_{K;K}X \equiv X$, then the duality of functors in this category is reflexive. Indeed, for any functor S and object C we have

$$DSC = \{S \to \Sigma_C\} = \{S \to T_{K,K}\Sigma_C\} = \{S\Omega_C \to T_{K,K}\} = \mathrm{Hom}(S\Omega_{CK}, K) = \overline{SC},$$

where $\bar{A} = \mathrm{Hom}(A, K)$. Hence

$$DDSC = DS\bar{C} = \overline{\overline{SC}} = SC.$$

For example, this is true in the category of locally compact topological abelian groups.

§ 2. The categories $\mathcal{G}_1$ and $\mathcal{G}_2$ of finite and finitely generated abelian groups. Let $\mathcal{C}$ be a concrete category such that a morphism $\varphi : C_1 \to C_2$ of this category is an isomorphism if and only if the mapping $|\varphi| : |C_1| \to |C_2|$ is one-to-one and onto. A functor S in the category $\mathcal{C}$ is called bounded (by the object A) if there exists an object A such that for every object B and element $\beta \in SB$ there are an element $a \in SA$ and a morphism $\varphi : A \to B$ such that $S\varphi(a) = \beta$.

Although the categories $\mathcal{G}_1$ and $\mathcal{G}_2$ are not D-categories, duality of functors can be defined in them. In the category $\mathcal{G}_1$, for any S and B there is an inclusion

$$\{S \to \Sigma_B\} \subset \mathrm{Hom}(SB, B \otimes B).$$

Therefore

$$DSB = \{S \to \Sigma_B\}$$

is a finite group. In the category $\mathcal{G}_2$ one can define the functor dual to a bounded functor by putting

$$DSB = \{S \to \Sigma_B\},$$

since

$$\{S \to \Sigma_B\} \subset \mathrm{Hom}(SA, A \otimes B).$$

We note that in the category $\mathcal{G}_2$ a functor S is bounded if and only if the groups $S^* Z_{p^h}$ (here $S^* A$ is the kernel of the natural homomorphism $A \otimes S\mathbb{Z} \to SA$) are trivial for all primes p, except for a finite set, and all sequences

$$S^* Z_p \to S^* Z_{p^2} \to S^* Z_{p^3} \to \cdots$$

stabilize. From this it is easy to deduce that in the category $\mathcal{G}_2$ the functor dual to a bounded functor is bounded. I do not know whether a similar assertion is true in any general category.

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* If $\mathcal{C}$ is a concrete category in which the functor Hom is defined, a functor S in it is called additive if the mapping

$$S : \mathrm{Hom}(A, B) \to \mathrm{Hom}(SA, SB)$$

is, for any $A, B \in \mathcal{C}$, a morphism of this category.

Let us note that the requirement of boundedness is essential in the definition of duality of functors in the category $\mathcal{G}_2$. For example, the functor V, where

$$VA = \mathrm{Hom}(Z_2 + Z_3 + Z_5 + \cdots, \mathrm{Tors}\, A),$$

$\mathrm{Tors}\, A$ being the torsion subgroup of the group A, is an additive functor $\mathcal{G}_2 \to \mathcal{G}_2$, while the group $\{V \to \Sigma_Z\}$ is infinitely generated.

§ 3. The operators D^2 and D^3 in the categories $\mathcal{G}_1$ and $\mathcal{G}_2$.

Theorem 1. *For every functor S in the category $\mathcal{G}_1$ the equality $D^2 S = S$ holds.*

Proof. Obviously,

$$\mathrm{Hom}(A, Z_{p^h}) = \mathrm{Hom}(A \otimes Z_{p^h}, Z_{p^h}).$$

Since the group $A \otimes Z_{p^h}$ is a finitely generated group all of whose elements have order dividing p^h, we have

$$\mathrm{Hom}(\mathrm{Hom}(A \otimes Z_{p^h}, Z_{p^h}) Z_{p^h}) = A \otimes Z_{p^h}.$$

Consequently,

$$T_{Z_{p^h}, Z_{p^h}} = \Sigma_{Z_{p^h}}.$$

Hence

$$DSZ_{p^h} = \{S \to \Sigma_{Z_{p^h}}\} = \{S \to T_{Z_{p^h}, Z_{p^h}}\} = \mathrm{Hom}(SZ_{p^h}, Z_{p^h})$$

(the last equality, formulated by us for D-categories, is also valid in the categories $\mathcal{G}_1$ and $\mathcal{G}_2$). Thus,

$$D^2 S(Z_{p^h}) = \mathrm{Hom}(\mathrm{Hom}(SZ_{p^h}, Z_{p^h}) Z_{p^h}).$$

Consequently,

$$D^2 S(Z_{p^h}) = T_{Z_{p^h}, Z_{p^h}} SZ_{p^h} = Z_{p^h} \otimes SZ_{p^h} = SZ_{p^h}.$$

For any additive functor in the category of abelian groups,

$$S(X + Y) = SX + SY.$$

The proposition is proved.

The analogous assertion in the category $\mathcal{G}_2$ is false even for bounded functors. For example, if Tors is the functor assigning to each finitely generated group its torsion subgroup, then D(Tors) is the identity functor.

Theorem 2. *Let S be a bounded functor in the category $\mathcal{G}_2$. Then for every finite group A, respectively for every group $A \in \mathcal{G}_2$, the equality $D^2SA = SA$, respectively the equality $D^3SA = DSA$, holds.*

Proof. The equality

$$\Sigma_{Z_{p^h}} = T_{Z_{p^h} Z_{p^h}}$$

also holds in the category $\mathcal{G}_2$ and is proved in the same way as in the category $\mathcal{G}_1$. Therefore the first part of the theorem is proved in the same way as Theorem 1. Further, it is easy to see that for any bounded functor S, a prime p, and sufficiently large h,

$$S(Z_{p^h}) = \underbrace{Z_{p^h} + \cdots + Z_{p^h}}_{n} + G_p,$$

where n and G_p do not depend on h.

The embedding $Z_{p^h} \to Z_{p^{h+1}}$ induces a homomorphism

$$\varphi_p : G_p \to G_p,$$

which also, for sufficiently large h, does not depend on h. Denote by G_p^* the subgroup of G_p consisting of elements stable with respect to φ_p. It is easy to verify that if $S = DS'$, then

$$SZ = \underbrace{Z + \cdots + Z}_{n} + \sum_p G_p^*$$

(whence it is seen that the G_p^* are nonzero only for a finite set of primes p). Hence it follows that $D^3SZ = DSZ$ for every bounded functor S, and therefore $D^3SA = DSA$ for every finitely generated group A.

§ 4. **Composition of functors.**

Theorem 3. *The equality $D(ST) = DS \circ DT$ holds: in the category $\mathcal{G}_1$ for arbitrary functors S and T; in the category $\mathcal{G}_2$ for arbitrary bounded functors S and T.*

Proof. As is known ([3]), the theorem on composition reduces to its special case:

$$D(\Omega_{AT}) = \Sigma_{ADT}.$$

In the category of groups it is enough to prove even less: the natural mapping

$$\Sigma_{ADT}\, B \to D(\Omega_{AT})B$$

is an epimorphism for all B (this follows from the fact that a mutually single-valued morphism in the category of groups is always an isomorphism). All these reductions can be obtained by transferring to our case the proof of Lemma 2 on p. 167 of the paper ([3]).

The first of the assertions after what has been said is obvious: since all functors are reflexive, $T = D^2T$. Obviously,

$$D(\Sigma_{ADT}) = \Omega^2_{AD}T = \Omega_{AT}$$

and, consequently,

$$D(\Omega_{AT}) = \Sigma_{ADT}.$$

Similarly we obtain that for a bounded functor T in the category $\mathcal{G}_2$ the equality

$$D(\Omega_A T)Z_{p^h} = \Sigma_A DTZ_{p^h}$$

holds. Here we use the simply proved fact: in the category $\mathcal{G}_2$ the composition of bounded functors is bounded. The groups $D(\Omega_A T)Z$ and DTZ are obtained from the groups $D(\Omega_A T)Z_{p^h}$ and DTZ_{p^h} by means of the procedure given in § 2. Hence it follows that, since

$$D(\Omega_A T)Z_{p^h} = \Sigma_A DTZ_{p^h}$$

for all p and h, we also have

$$D(\Omega_A T)Z = \Sigma_A DTZ.$$

§ 5. The category of all abelian groups.

Theorem 4. In the category $\mathcal{G}$ the equality $D^2SA = DSA$ holds:

(I) For any finite group A, if the functor S carries finite groups into finite groups.

(II) For any finitely generated group A, if the functor S is bounded by a finitely generated group.

(III) For any group A, if the functor S is bounded by a finite group.

Proof. To prove (I), it is enough to show that

$$D^3SZ_{p^h} = DSZ_{p^h}$$

for arbitrary prime p and natural h. Let us note that for any group B there is an inclusion

$$\Sigma_{Z_{p^h}} B \subset T_{Z_{p^h}, Z_{p^h}} B.$$

Therefore

$$DSZ_{p^h} = \{S \to \Sigma_{Z_{p^h}}\} \subset \{S \to T_{Z_{p^h}, Z_{p^h}}\} = \mathrm{Hom}(SZ_{p^h}, Z_{p^h}).$$

Hence the order of the group DSZ_{p^h} does not exceed the order of the group SZ_{p^h}. On the other hand, the mapping

$$\chi^S_{Z_{p^h}} : DSZ_{p^h} \to D^3SZ_{p^h}$$

is a monomorphism. Consequently,

$$D^3SZ_{p^h} = DSZ_{p^h}.$$

We do not give the full proofs of assertions (II) and (III). The main difficulty in the proof of (II) is that, from the fact that the functor S is bounded by a finitely generated group, it does not follow, in general, that this property holds for DS. It turns out to be possible to extract from this property certain consequences which are preserved under passage to the dual functor and which suffice for the group DSZ to be determined by the groups DSZ_{p^h} analogously to how this was done in § 2. Conversely, the proof of assertion (III) is based on the fact that if the functor S is bounded by a finite group, then the functor DS is also bounded by a finite group.

Theorem 5. In the category $\mathcal{G}$ the equality

$$D(ST)A = DS(DTA)$$

holds:

(I) If the functors S and T are bounded by finitely generated groups, and the group A is finitely generated.

(II) If the functors S and T are bounded by finite groups, the group A is arbitrary.

Proof. Assertion (I) will be proved if we prove that for finitely generated groups A and B the equality

$$\Sigma_A DTB = D(\Omega_A T)B$$

holds. For any functor U bounded by a finitely generated group, we have:

$$DUZ_{p^h} = \mathrm{Hom}(UZ_{p^h}, Z_{p^h})$$

(this follows from the fact that

$$\Sigma_{Z_{p^h}} C = T_{Z_{p^h}, Z_{p^h}} C$$

for finitely generated C). Therefore

$$D(\Omega_A T)Z_{p^h} = \mathrm{Hom}(\mathrm{Hom}(A, TZ_{p^h}), Z_{p^h}) = A \otimes \mathrm{Hom}(TZ_{p^h}, Z_{p^h}) = \Sigma_A DTZ_{p^h}.$$

The equality

$$\Sigma_A DTZ = D(\Omega_A T)Z$$

is obtained in the same way as in the proof of Theorem 3.

Assertion (II) follows from the fact that the property of a functor being bounded by a finite group is preserved under passage to the dual. In general, one may note that if the functors S, T, and $D(ST)$ are bounded by finitely generated groups, then the equality

$$D(ST) = DS \circ DT$$

holds.

§ 6. Generalizations. The results of the note are also valid in some other categories. In any case, the methods presented are applicable only to categories in which the objects are abelian groups and the morphisms are homomorphisms of these groups. For example, in the category of Banach spaces we obtain that for any functor S, bounded by a finite-dimensional

space, the equality $D^3S = DS$ holds; for two such functors S and T the equality $D(S \circ T) = DS \circ DT$ holds. If S is endowed with a Hilbert-space structure, then $DSA = \mathrm{Hom}(SR, A)$, where R is the line.

§ 7. In conclusion, on one unsolved problem. Let $\mathcal{C}$ be an autonomous category ([4]), i.e., a concrete category in which the functors Σ_A and Ω_A, possessing the usual properties, are defined. We shall say that in this category a duality of functors is defined if

1. A contravariant functor $D : \mathcal{F} \to \mathcal{F}$ is given, where $\mathcal{F}$ is the category of additive covariant functors of the category $\mathcal{C}$.

2. For any functors $S, T \in \mathcal{F}$ and a mapping $\varphi : DS \to T$, a mapping $\rho_{S,T}\varphi : DT \to S$ is given, with

$$\rho_{T,S} \circ \rho_{S,T}\varphi = \varphi.$$

3. Let X be an object of the category, and S and T functors. Suppose that for each $x \in X$ a mapping $\varphi_x : DS \to T$ is given such that, for any $Y \in \mathcal{C}$, the correspondence $x \to (\varphi_x)_Y$ defines a morphism

$$X \to \mathrm{Hom}(DSY, TY).$$

   Then the correspondence $x \to (\rho_{S,T}\varphi_x)_Y$, for all $Y \in \mathcal{C}$, defines a morphism

$$X \to \mathrm{Hom}(DTY, SY).$$

4. $D\Sigma_A = \Omega_A$ and $D\Omega_A = \Sigma_A$ for all A.

Question. Under what conditions is this duality reflexive?

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Note: Figure translations are in progress. See original paper for figures.