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Abstract

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ON THE DIVERGENCE OF TRANSFORMATIONS OF DIFFERENTIAL EQUATIONS TO NORMAL FORM

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§ 1. Let $\varphi_1(X), \dots, \varphi_n(X)$ be power series in $x_1, \dots, x_n$ without constant terms, convergent in some neighborhood of the point $X = 0$. Then $X = 0$ is a singular point of the system of differential equations

$$dx_i/dt = \varphi_i(X), \qquad i = 1, \dots, n. \tag{1}$$

Let, further, $\xi_1(Y), \dots, \xi_n(Y)$ be such power series in $y_1, \dots, y_n$, without constant terms, that the transformation

$$x_i = \xi_i(Y), \qquad i = 1, \dots, n, \tag{2}$$

takes system (1) into the normal form (see below)

$$dy_i/dt = \psi_i(Y), \qquad i = 1, \dots, n. \tag{3}$$

Thus, A. Poincaré (^{1}) proved that there exists a biholomorphic transformation (2) at zero which brings system (1) to the form

$$dy_i/dt = \lambda_i y_i, \qquad i = 1, \dots, n, \tag{4}$$

if the eigenvalues $\lambda_1, \dots, \lambda_n$ of the matrix $\|\partial\varphi_i/\partial x_j\|_0$ satisfy certain stringent conditions. The coefficients of all power series under consideration are assumed to be complex numbers. A power series will be called *divergent* if there is no spherical neighborhood of zero in which it converges. A *divergent transformation* (2) is defined analogously.

Write system (3) in the form

$$dy_i/dt = y_i g_i(Y) \equiv y_i \sum_{Q \in N_i} g_{iQ} Y^Q, \qquad i = 1, \dots, n, \tag{5}$$

where $Q = (q_1, \dots, q_n)$, $Y^Q = y_1^{q_1} \cdots y_n^{q_n}$; $N_i = \{Q :$ the q_j are integers, $q_j \geq 0$, $j \neq i$; $q_i \geq -1$; $q_1 + \cdots + q_n \geq 0\}$, $i = 1, \dots, n$. Denote $N = N_1 \cup \dots \cup N_n$ and $\Lambda = (\lambda_1, \dots, \lambda_n)$, the vector of eigenvalues of the matrix $\|\partial \varphi_i / \partial x_j\|_0$.

System (3) is called a *normal form* if the matrix $\|\partial \psi_i / \partial y_j\|_0$ is a Jordan normal form, and, when system (3) is written in the form (5), $g_{iQ} = 0$ if $(Q, \Lambda) \neq 0$. Every formal system (1), by means of an invertible formal transformation (2), can be reduced to its normal form ([2]). Generally speaking, the normal form and the transformation to it are not determined uniquely by the original system.

Since the time of Poincaré, many works have been devoted to solving the following problems. For which normal forms (3) does the analyticity of the original system (1) imply the analyticity of the transformation (2)? For which normal forms (3) do there exist analytic systems (1) that are reduced to the normal form by a divergent transformation?

In the paper ([3]), conditions on the normal form (3) are given that are sufficient for convergence of the transformation (2). All previously known cases of convergence satisfy these conditions. In the present work, weaker

conditions that are already necessary and sufficient for the convergence of the transformations to normal form. The divergence of these transformations is known at the present time in the following three cases:

a) $n = 2$ and system (1) is equivalent to a linear equation in x_2 ([4]);

b) when anomalously "small denominators" (Q, Λ) appear, i.e., when $|(Q, \Lambda)|$ tends too rapidly to zero as $\sum q_i \to \infty$, ([5]); ([6], §28);

c) for real Hamiltonian systems with two degrees of freedom, in which all λ_i are purely imaginary, the basic ones among them are linearly independent over the integers, and the normal form is nondegenerate ([7]).

§ 2. Let us represent the numbers λ_i by points in the complex plane; two cases are possible:

1. There exists a straight line passing through zero such that on one side of it there are no points λ_i.
2. On both sides of every straight line passing through zero there lie points λ_i.

Let us consider case 1 in more detail. Let L be a straight line with the indicated properties and such that on it lies the smallest possible number l of points λ_i $(0 \leq l \leq n)$. Introduce such a numbering of the variables Y that $\lambda_1, \dots, \lambda_l$ lie on L, while $\lambda_{l+1}, \dots, \lambda_n$ lie on one side of L. Let τ_j be the distance of the point λ_j

from the line L, and let the vectors $R, \dots, S$ form a basis of the linear subspace in R^n orthogonal to all solutions $Q \in N$ of the equation

$$\sum_1^n q_i \tau_i = 0.$$

Obviously, $r_i = \dots = s_i = 0$ for $i \leq l$. With the introduced numbering, the normal form (5) has the form

$$dy_i/dt = y_i g_i, \quad i = 1, \dots, l;$$

$$dy_j/dt = \sum_{k=l+1}^n a_{jk} y_k + \eta_j(y_1, \dots, y_n), \quad j = l+1, \dots, n, \tag{5'}$$

where $y_i g_i$ and a_{jk} are series in $y_1, \dots, y_l$, and η_j contains no terms linear in $y_{l+1}, \dots, y_n$.

Condition A on the normal form (3), in the notation (5) and (5′), depending on Λ, consists in the following:

A$_1$. If Λ belongs to case 1, then there exists a power series μ in $y_1, \dots, y_l$ such that $g_i = \lambda_i \mu$, $i = 1, \dots, l$; if $\lambda_1, \dots, \lambda_l$ are pairwise commensurable, then $g_{l+1}, \dots, g_n$ are arbitrary; but if among $\lambda_1, \dots, \lambda_l$ there is a pair of incommensurable ones, then there also exist power series $\rho, \dots, \sigma$ in $y_1, \dots, y_l$ such that the matrix

$$\left\| a_{jk} - \delta_{jk}(\lambda_j \mu + r_j \rho + \dots + s_j \sigma) \right\|$$

is nilpotent, where δ_{jk} is the Kronecker symbol.

A$_2$. If Λ belongs to case 2, then there exist power series μ and ν in Y such that

$$g_i = \lambda_i \mu + \overline{\lambda_i} \nu, \quad i = 1, \dots, n.$$

If some normal form of system (1) satisfies condition A, then every normal form of system (1) satisfies this condition. Suppose Λ belongs to case 1 and $\lambda_1, \dots, \lambda_l$ are commensurable; then there exists $\varepsilon > 0$ such that for every $Q \in N$ either $(Q, \Lambda) = 0$, or $|(Q, \Lambda)| > \varepsilon$; consequently, condition α is satisfied (see below). Indeed, then $\omega_k > \varepsilon$, and

$$\sum_1^\infty 2^{-k} \ln \omega_k^{-1} < \sum_1^\infty 2^{-k} \ln \varepsilon^{-1} = \ln \varepsilon^{-1} < \infty.$$

Finally, if $l = 0$, then A$_1$ contains no restrictions on $g_1, \dots, g_n$. This last case was studied in detail by A. Dulyak (^{11}).

Condition α. Put $\omega_k = \min |(Q,\Lambda)|$ over $Q \in N$, $\sum q_i < 2^k$, and $(Q,\Lambda) \neq 0$. Then

$$\sum_{1}^{\infty} 2^{-k} \ln \omega_k^{-1} < \infty. \tag{6}$$

Condition α'.

$$\overline{\lim_{k\to\infty}}\, 2^{-k} \ln \omega_k^{-1} < \infty.$$

This condition is equivalent to the inequality

$$|(Q,\Lambda)| > c_1 \exp\left(-c_2 \sum q_i\right)$$

for $Q \in N$ with some $c_1, c_2 > 0$.

Condition α' is weaker than α, since the convergence of the series (6) implies the boundedness of its terms, i.e., condition α' is satisfied.

Theorem 1. *If, for system (1), Λ satisfies condition α, the normal form (5) satisfies condition A, and the φ_i are analytic at zero, then there exists a transformation of system (1), analytic at zero, to some normal form.*

The proof is quite complicated and is carried out by the "generalized Newton method." For the simplest case, a proof of this kind was given by V. A. Pliss ([8]).

The question of the divergence of transformations is nontrivial only for normal forms (3) in which the $\psi_i(Y)$ converge in some neighborhood of zero. Otherwise, an invertible analytic transformation (2) leads to (3) only those systems (1) in which some of the φ_i necessarily diverge.

Theorem 2. *If the normal form (3) is such that one or both of the conditions α' and A are not satisfied, and the ψ_i are analytic at zero, then there exists an analytic system (1) for which system (3) is a normal form and every transformation of it to any normal form is divergent.*

The proof consists in indicating a method for constructing, from the normal form (3), a system (1) with the required properties. If system (1) is brought to normal form by a divergent transformation in which arbitrarily specified coefficients (the cause of nonuniqueness) are equal to zero, then every transformation of system (1) to normal form will be divergent. If condition A is violated, then, in order to construct one such system (1) from the given normal form (3), one uses power transformations ([9]) and the geometry of degree exponents associated with them ([2,10]). If condition α' is violated, then system (1) is constructed in the same way as in ([5]) and ([6]), § 28. Theorem 2 contains all previously known cases of divergence of transformations (2) in the class of arbitrary (not necessarily Hamiltonian) initial systems (1).

Thus, the questions posed above have been resolved for all normal forms except those for which Λ satisfies condition α' but does not satisfy α, while the remaining coefficients of the normal form satisfy condition A.

§ 3. We now show what Theorem 1 gives for Hamiltonian systems

$$dx_i/dt = \partial H/\partial x_{i+m}, \qquad dx_{i+m}/dt = -\partial H/\partial x_i, \quad i = 1, \dots, m, \tag{7}$$

where $H(x_1, \dots, x_{2m})$ is a power series without linear terms. Here $n = 2m$, $\lambda_i = -\lambda_{i+m}$, $\Lambda = (\lambda_1, \dots, \lambda_{2m})$. If $\lambda_1, \dots, \lambda_m$ are linearly independent over the integers, then, as Birkhoff showed (^{12}), system (7) has the Hamiltonian normal form

$$\partial y_i/\partial t = \partial h/\partial y_{i+m}, \qquad \partial y_{i+m}/\partial t = -\partial h/\partial y_i, \quad i = 1, \dots, m, \tag{8}$$

where h is a power series in the products $y_i y_{i+m}$ $(i = 1, \dots, m)$.

The Hamiltonian normal form (8) satisfies condition A only in two cases:

1. All the points $\lambda_1, \dots, \lambda_m$ lie on one straight line passing through the origin of the complex plane, and h is a series in powers of
$$u = \sum_1^m \lambda_i y_i y_{i+m}.$$
2. The points $0, \lambda_1, \dots, \lambda_m$ do not lie on one straight line, and h is a series in powers of two variables
$$u = \sum_1^m \lambda_i y_i y_{i+m} \quad \text{and} \quad v = \sum_1^m \overline{\lambda}_i y_i y_{i+m}.$$

In cases 1 and 2 the conditions A_1 with $l = n$ and A_2, respectively, are satisfied. If, moreover, Λ satisfies condition α, then, by Theorem 1, there exists an analytic change of variables reducing (7) to some normal form. Then, as Rüssmann proved ^{13}, there exists a canonical analytic transformation (2) reducing (7) to the Hamiltonian normal form (8). Finally, in cases 1 and 2 there are no conditions on h, i.e. they coincide with the assertion of Birkhoff' s theorem only in the following two subcases: the first, $m = 1$, was investigated by Poincaré and Lyapunov, and the second, $m = 2$, $\mathrm{Im}(\lambda_1/\lambda_2) \neq 0$, was investigated by J. Moser ^{14}. In these subcases condition α is satisfied.

In conclusion, let us consider the case $m = 2$, λ_1/λ_2 real irrational. If the original system (7) is real, λ_1/λ_2 satisfies some "small-denominator" condition, and h is not a series only in powers of $u = \lambda_1 y_1 y_3 + \lambda_2 y_2 y_4$, then, as V. I. Arnol' d proved ^{15}, zero is a stable singular point of the system (7). If, however, h is expressed as a series in powers of u, i.e. belongs to case 1, and λ_1/λ_2 satisfies the "small-denominator" condition, then conditions A and α are satisfied, and, by Theorem 1, system (7) is reduced to some normal form by an analytic change of variables*. Hence the stability of zero follows in this exceptional case as well.

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* *Proof correction note.* After the paper had been submitted for printing, a proof of this appeared [16].

Note: Figure translations are in progress. See original paper for figures.

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