Controllability in a Hilbert space

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Abstract

Full Text

Preamble

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Let H be a separable Hilbert space, and let A be a linear bounded operator mapping H into H. Consider the equation:

$$\frac{dx}{dt} = Ax + bu, \quad (1)$$

where u is a scalar function of time ($t < \infty$) called the control. Let us initially assume that $u(t) \in L_2$. We shall define the function $x = x(t)$ (where t is a real argument and $x(t) \in H$) as a solution to equation (1) given $u = u(t)$ on an interval if the equality holds almost everywhere within that interval.

^ ± =Ax(t) + bu(t). dt

It is easily verified that the solution to equation (1) with the initial condition $x(0) = x_0$ is given by the Cauchy formula:

$$x(t) = [\exp At]x_0 + \int_0^t [\exp A(t-\tau)]bu(\tau)d\tau.$$

By definition, we shall refer to the generalized solution of the equation as the function $x(t) = \exp(At)x_0 + \int_0^t [\exp A(t-\tau)]bd\sigma(\tau)$, where $d\sigma(\tau)$ is the Stieltjes differential of a function of bounded variation σ. In particular, if $\sigma(t)$ is a differentiable function, then $d\sigma(t) = u(t)dt$. We define the dynamical system described by equation (1) as controllable if, for any point in the space H, there exists a function of bounded variation (an admissible control) and a finite time $T > 0$ such that the state reaches that point. Here, the state is determined by formula (2), and x_0 denotes the initial point in the space H. It is well known that in the case where the space H is n-dimensional, the linear independence of the vectors $b, Ab, A^2b, \ldots, A^{n-1}b$ ensures the controllability of the system. Naturally, the question arises as to what role the sequence $b, Ab, A^2b, \ldots, A^{nb}, \ldots$ plays when the space H is infinite-dimensional. We shall call a system a basis of the space H if every element $x \in H$ can be represented (not necessarily uniquely) in the form of a series with coefficients c_n.

F. A. SHOLOKHOVICH Theorem: System (1) is controllable if the set···

E = {b,Ab,A 2 b,···, A m b,···}

is a basis in the space. **Proof.** Let x be an arbitrary point in the space R. For this point, there exists an admissible control u and a number T such that

$$x(T) = [\exp AT]x_0 + \int_0^T [\exp A(T-\tau)]Bu(\tau)d\tau = q.$$

From this, we obtain $[\exp(-AT)]x(T) - x_0 = \int_0^T [\exp(-A\tau)]Bu(\tau)d\tau$. The integrand can be represented as a series. We shall now demonstrate the validity of term-by-term integration for this series. By applying Theorem 3.7.5 and Corollary 1 of Section IV (see [**?**], pp. 95 and 78), we obtain:

$$\int_0^T \left\| \sum_{n=0}^{\infty} \frac{(-A\tau)^n}{n!} Bu(\tau) \right\| d\tau \le \sum_{n=0}^{\infty} \frac{\|A\|^n T^n}{n!} \|B\| \int_0^T \|u(\tau)\| d\tau < \infty.$$

< { sup

$Var.[a(x)]$. As $n \to \infty$, the latter expression tends to zero, which proves the validity of term-by-term integration. Performing term-by-term integration on the left side of equality (3), we obtain:

$$\int_0^T t^k da(x) = Ab^k \int_0^T d\sigma(x) + (-1)^k b^k \int_0^T d\sigma(x)$$

The theorem is proved. It is easy to establish that the infinite dimensionality of the linear span of a set implies the linear independence of any finite number of elements taken from this set. Now, let the set serve as a basis for the space H and, furthermore, let the following requirement be satisfied (the condition for an arbitrary H): the expansion is uniquely determined.

The condition will be satisfied, for example, if in the coordinate Hilbert space l_2 we take the vector $\{1, 0, 0, \ldots, 0, \ldots\}$ as x and define the operator A using a matrix. Obviously, $Ax_1 = \{0, 1, 0, \ldots\}$, $Ax_2 = \{0, 0, 1, 0, \ldots\}$, and so on. This condition is quite restrictive. When it is satisfied, the operator A, for instance, cannot have eigenvalues other than zero. Indeed, let $Ax = \lambda x$; for simplicity, assume that λ is a real number. Represent the eigenvector of λ in the form

$x = \sum \alpha_i e_i$. By the continuity of the operator, we have $Ax = \sum \alpha_i Ae_i$. Using the equality $\lambda x = Ax$, we obtain $\lambda \sum \alpha_i e_i = \sum \alpha_i e_{i+1}$. From this, it follows that $\alpha_0 = 0$, and consequently, $\alpha_1 = 0$. From this equality, it further follows that $\alpha_2 = 0$, and in general, $\alpha_n = 0$ for $n = 0, 1, 2, \ldots$. Thus, $x = 0$, contrary to the assumption. We shall now show that in an infinite-dimensional space, Theorem 1 is not reversible if condition C is satisfied. Let some point be transferred by the control $u(t)$ to the initial point θ in time T. Then equality (3) holds, and consequently, equality (4) holds as well. Suppose that $x(T) = 0$. Using equality (4), we obtain:

$$\int_0^T t^m da(t) = (-1)^m \frac{m!}{b^m} x_m \quad (m = 0, 1, 2, \ldots)$$

Conversely, if there exist a number T and a function of bounded variation $a(t)$ such that equalities (5) hold, then the point is transferred to point θ by the control $u(t)$ in time T.

SHOLOKHOVICH. Thus, the question of the existence of a control for a given point that transfers the point to the equilibrium position in finite time is equivalent to the question of the solvability of the power moment problem (5). We perform a change of variables by setting $t = \tau/T$ and denote $g(t) = a(Tt)$. From equality (5), we obtain:

$$\int_0^1 t^m dg(t) = (-1)^m \frac{m!}{b^m T^m} x_m$$

We utilize the following theorem by Hausdorff ([?], p. 209): In order for there to exist a function of bounded variation $g(t)$ such that···

1 m̂ dg(t):-ll m (til = • 0, 1, 2, . . .) ,

Theorem and Proof of Reachability

It is necessary and sufficient that for $n = 1, 2, \ldots$, the binomial coefficients and the n-th differences of the sequence satisfy the specified conditions. Let C be a certain positive constant. We denote the n-th differences of the sequence as $\Delta^n a_k$.

Theorem

If the condition is satisfied starting from a certain index n, and the numbers in the expansion are non-negative (or non-positive), then at these points, the system cannot be transferred to the origin from the initial state by any admissible control $u(t)$ in finite time according to the dynamics of equation (2). In other words, such a point does not belong to the reachability set of the initial point.

Proof

Indeed, let us assume that $t > 0$ and consider the properties of the control trajectory. Suppose there exists an admissible control that brings the system to the target state. By analyzing the sequence of differences and the signs of the coefficients in the expansion, we can demonstrate a contradiction regarding the finite-time reachability. Specifically, if the coefficients maintain a consistent sign (non-negative or non-positive) beyond a certain threshold, the integral representation of the trajectory under equation (2) precludes the state from reaching the origin within a finite interval.

Let

$\hat{T}_m = (-1)^m \hat{T}^{(m)}$ (8)

For an arbitrary, fixed number k, the formula $C_m^k = \frac{m(m-1)...(m-k+1)}{k!}$ holds. Taking into account the conditions of the theorem and formula (8), we obtain

Consequently, as $m \to \infty$, we have

m 2c $/A$ M -|i*|->«>.

Condition (7), which is necessary for the solvability of the moment problem, is satisfied. The theorem is proved.

In particular, points whose expansions contain only a finite number of non-zero coefficients (where at least one of these coefficients has a positive index) belong to the reachable set of the origin. We define the dynamical system described by the equation as stabilizable if, for any point x_0, there exists an admissible control $u(t)$ such that the corresponding solution to the equation satisfies the property:

$$\lim_{t \to \infty} x(t, x_0, u) = 0.$$

Theorem: If the real-valued function $\| \exp(At) \|$ is bounded and the set $\{b, Ab, A^2b, \dots, A^n b, \dots\}$ forms a basis for the space H, then the dynamical system corresponding to the equation is stabilizable.

Proof: By transforming the Cauchy formula, we obtain:

$$x(t) = \exp(At) \left(x_0 + \int_0^t \exp(-A\tau) b u(\tau) d\tau \right)$$

We must establish the existence of a control $u(t)$ that ensures stabilization. Applying term-by-term integration here, we obtain:

$$u_0(x) - \int_0^t \tau A b u(\tau) d\tau + \dots + \frac{(-1)^m}{m!} \int_0^t \tau^m A^m b u(\tau) d\tau + \dots \quad (11)$$

Consider one of the expansions of the element x_0 in the basis $\{b, Ab, \dots, A^m b, \dots\}$ (where uniqueness of the expansion is assumed):

$$x_0 = \sum_{m=0}^{\infty} x_m^0 A^m b$$

To ensure that condition (11) is satisfied, it is sufficient to select a function $u(\tau)$ that solves the moment problem:

$$\int_0^{\infty} \tau^m u(\tau) d\tau = (-1)^m x_m^0 m! \quad (m = 0, 1, 2, \dots).$$

As is well known ([**?**], p. 103, Theorem 3.11), this problem has an infinite set of solutions for arbitrary right-hand sides. Let $u(t)$ be one such solution. Then

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From the formula $\|x(t)\| \leq \|x(0)\| \, \|\exp(At)\|$, by virtue of the given condition and the boundedness of $\|\exp(At)\|$, we obtain

$$\lim_{t \to \infty} \|x(t)\| = 0.$$

The theorem is proved. The author expresses his gratitude to N. N. Krasovskii for discussing this work.

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Figures

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ON CONTROLLABILITY IN HILBERT SPACE

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Let H be a separable Hilbert space; A — a linear bounded operator mapping H to H, $b \in H$.

Consider the equation

$$\frac{dx}{dt} = Ax + bu, \tag{1}$$

where $x \in H$, $u(t)$ — a scalar function of time t $(0 \leqslant t < \infty)$, called the control. Let us assume first that $u(t) \in L_2$.

The function $x = x(t)$ (t — real argument, $x(t) \in H$) will be called a solution of equation (1) for $u = u(t)$ in the interval $[0, T]$, if almost everywhere in this interval the equality holds

$$\frac{dx}{dt} = Ax(t) + bu(t).$$

It is not difficult to check that the solution $x(t)$ of equation (1) with the initial condition $x(0) = x_0$ has the form (Cauchy formula)

$$x(t) = [\exp At]\, x_0 + \int_0^t [\exp A(t-\tau)]\, bu(\tau)\, d\tau.$$

We will by definition call the generalized solution of equation (1) the function

$$x(t) = [\exp At]\, x_0 + \int_0^t [\exp A(t-\tau)]\, b\, d\sigma(\tau), \tag{2}$$

where $d\sigma(\tau)$ — the Stieltjes differential of a function of bounded variation $\sigma(t)$ and, in particular, if $\sigma(t)$ — is a differentiable function, $d\sigma(t) = u(t)dt$.

We call the dynamical system described by equation (1) *controllable*, if for any point $x_0 \in H$ there exists such a function $\sigma_1(t)$ of bounded variation (admissible control) and such a finite time — $t = T$ $(T > 0)$, that $x(T) = \theta$. Here $x(t)$ is defined by formula (2), and θ denotes the initial point of the space H.

It is known that in the case when the space H is n-dimensional, the linear independence of the vectors $b, Ab, A^2b, \ldots, A^{n-1}b$ ensures the controllability of the system.

A natural question arises, what role is played by the sequence $b, Ab, A^2b, \ldots, A^mb, \ldots$, if the space H is infinite-dimensional.

We call the system $\{e_m\}_0$ a *basis* of the space H, if every element $x \in H$ can be represented (not necessarily uniquely) in the form

$$x = \sum_{m=0}^{\infty} \alpha_m e_m,$$

where α_m — are numbers.

Figure 1: Figure 1

Theorem 1. *If system* (1) *is controllable, then the set*

$$E = \{b, Ab, A^2b, \ldots, A^mb, \ldots\}$$

is a basis in the space H.

Proof. Let x_0 be an arbitrary point of the space H. For this point there exists such an admissible control $\sigma(t)$ and such a number T, that

$$x(T) = [\exp AT]x_0 + \int_0^T [\exp A(T-\tau)]b\,d\sigma(\tau) = 0.$$

From this we obtain

$$\int_0^T [\exp(-A\tau)]b\,d\sigma(\tau) = -x_0. \tag{3}$$

The integrand is represented as a series. We will show the possobility of term-by-term integration of this series.

Using Theorem 3. 7. 5 and Corollary 1 p. IV (see [1], pp. 95 and 78), we obtain

$$\left\|\int_0^T [\exp(-A\tau)]b\,d\sigma(\tau) - \sum_{m=0}^{n}\int_0^T (-1)^m \frac{\tau^m}{m!}A^{mb}\,d\sigma(\tau)\right\| =$$

$$= \left\|\int_0^T \left(\sum_{m=n+1}^{\infty}(-1)^m \frac{\tau^m}{m!}A^{mb}\right)d\sigma(\tau)\right\| \leqslant$$

$$\leqslant \left\{\sup_{0<n<1}\left\|\sum_{m=n+1}^{\infty}(-1)^m \frac{\tau^m}{m!}A^{mb}\right\|\right\}\operatorname{Var}[\sigma(\tau)] \leqslant$$

$$\leqslant \left\{\sup_{0<n<1}\sum_{m=n+1}^{\infty}\frac{(\tau\|A\|)^m}{m!}\|b\|\right\}\operatorname{Var}[\sigma(\tau)].$$

As $n \to \infty$ the last expression tends to zero, which proves the possobility of term-by-term intergation. Performing term-by-term intergation in the left part of equalty (3), we obtain

$$-x_0 = b\int_0^T d\sigma(\tau) - Ab\int_0^T \tau\,d\sigma(\tau) + \ldots + (-1)^m A^m b\int_0^T \frac{\tau^m}{m!}\,d\sigma(\tau) + \ldots \tag{4}$$

The theorem is proved.

It is easy to establish that from the enfinive dimensionality of dimenionality of the linear span of the set E follows the linear independence of amy elements of this set taken in a finite number.

Let now the set E be a basis of the space H and, morwence, the following requirement be fulfolned (*condition C*); for an arbibory $x \in H$ the the expansion

$$x = \sum_{m=0}^{\infty} a_m a^{mb}$$

is determined uniquuely.

Figure 2: Figure 2

Condition C will, for example, be fulfilled, if in the coordinate Hilbert space l_2 we take as b the vector $\{1, 0, 0, \ldots\}$, and the operator A is given with the help of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & . & . & . \\ 1 & 0 & 0 & 0 & . & . & . \\ 0 & 1 & 0 & 0 & . & . & . \\ 0 & 0 & 1 & 0 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{pmatrix}.$$

Obviously, $Ab = \{0, 1, 0, \ldots\}$, $A^2b = \{0, 0, 1, 0, \ldots\}$ and so on.

Condition C is very restrictive. With its fulfillment the operator A, for example, cannot have eigenvectors, distinct from zero.

Indeed, let $Ax = \lambda x$, $\lambda \neq 0$, $x \neq \theta$. For simplicity we assume, that λ — a real number. The eigenvector x of the number λ we represent in the form

$$x = \sum_{m=0}^{\infty} \alpha_m A^m b.$$

By virtue of the continuity of operator A we have

$$Ax = \sum_{m=0}^{\infty} \alpha_m A^{m+1} b.$$

Using the equality $\lambda x = Ax$, we obtain

$$\sum_{m=0}^{\infty} \alpha_m \lambda A^m b = \sum_{m=0}^{\infty} \alpha_m A^{m+1} b.$$

From here $\alpha_0 \lambda = 0$, consequently, $\alpha_0 = 0$. From the same equality it follows further, that $\alpha_1 = 0$, $\alpha_2 = 0$ and in general $\alpha_m = 0$ for $m = 0, 1, 2, \ldots$ Thus, $x = \theta$ contrary to the assumption.

Let us show now, that in the infinite-dimensional space H theorem 1 is irreversible, if condition C is fulfilled.

Let some point $x_0 \in H$ be transferred by control $\sigma(t)$ in time T to the initial point θ. Then equality (3) is fulfilled and, consequ, consequently, equality (4).

Let us assume, that

$$x_0 = \sum_{m=0}^{\infty} x_m^0 A^m b.$$

With the help of equality (4) we obtain

$$\int_0^T v^n d\bar{\sigma}(t) = (-1)^{n+1} m!\, x_m^1 \quad (m = 0, 1, 2, \ldots). \tag{5}$$

Conversely, if there exist a number T and a function $o(t)$ of brounded variation, such, that equalities (5) take place, then the point

$$x_0 = \sum_{m=0}^{\infty} x_m^0 A^m b$$

is transferred by control $\sigma(t)$ in time T to the point θ.

Figure 3: Figure 3

Thus, the question of the existence for a given point x_0 of a control that transfers this point in finite time to the position θ is equivalent to the question of the solvability of the power moment problem (5). Let's make a change of variable, setting $\tau = Tt$ and denote $g(t) = \sigma(Tt)$. From equality (5) we obtain

$$\int_0^{11} t^m\, dg(t) = (-1)^{m+1} \frac{x_m^0 m!}{T^m} \quad (m = 0, 1, 2, \ldots). \tag{6}$$

We use the following Hausdorff theorem ([2], p. 209). For there to exist a function of bounded variation $g(t)$ such that

$$\int_0^{11} t^m\, dg(t) = \mu_m \quad (m = 0, 1, 2, \ldots),$$

it is necessary and sufficient that

$$\textstyle\sum_{k=0,}^{m} C_m^k |\Delta^{m-k}\mu_k| \leq M \quad (m = 0, 1, 2, \ldots), \tag{7}$$

where C_m^k are binomial coefficients; M is some positive constant, and $\Delta^{\circ}\mu_k$ denote the n-th differences for the sequence $\{\mu_k\}$.

Theorem 2. If condition C is satisfied and, starting from a certain number $N > 0$, all numbers x_m^0 in the expansion

$$x_0 = \sum_{m=0}^{\infty} x_m^0 A^m \mathbf{b}$$

are non-negative (or non-positive), with $x_n \neq 0$, then the point x_0 ($x_0 \neq \theta$) cannot be transferred by any admissible control $\sigma(t)$ by virtue of equation (2) in finite time to the position θ (i.e. the point x_0 does not belong, as they say, to the reachability set of the initial point θ).

Theorem 2. If condition C is satisfied and, starting from a certain number $N > 0$, all numbers x_m^0 in the expansion

$$x_0 = \sum_{m=0}^{\infty} x_m^0 A^m \mathbf{b}$$

are non-negative (or non-positive), with $x_n \neq 0$, then the point x_0 ($x_0 \neq \theta$) cannot be transferred by any admissible control $\sigma(t)$ by virtue of equation (2) in finite finite time to the position θ (i.e. the point x_0 does not belong, as they say, to the reachability set of the initial point θ).

Proof. Indeed, let $x_m^0 \geq 0$ for $m \geq N > 0$ and $x_n^0 > 0$. Let us set

$$\mu_m = (-1)^{m+1} \frac{x_m^0 m!}{T^m} \tag{8}$$

for an arbitrary but fixed number $T > 0$. The formula ($k < m$) holds

$$\Delta^{m-k}\mu_k = \mu_k - C_{m-k}^1 \mu_{k+1} + C_{m-k}^2 \mu_{k+2} + \cdots + (-1)^{m-k}\mu_m.$$

For $k = N$, taking into account the condition of the theorem and formula (8), we obtain

$$|\Delta^{m-N}\mu_n| \geq |\mu_n| > 0.$$

Therefore, as $m \to \infty$ we have

$$\sum_{k=0}^{m} C_m^k |\Delta^{m-k}\mu_k| \to \infty.$$

Condition (7), necessary for the solvability of the moment problem, is not fulfilled. The theorem is proved.

Figure 4: Figure 4

In particular, the points x_0, in the expansion of which only a finite number of the numbers x_m^0 are non-zero (among these numbers there is at least one with a positive index), do not belong to the reachable set of the point θ. We will call the dynamical system described by equation (1) *stabilized* if for any point $x_0 \in H$ there exists an admissible control $\bar{\sigma}(t)$ such that the corresponding solution $x(t, x_0, \bar{\sigma})$ of equation (1) has the property:

$$\lim_{t\to\infty} x(t, x_0, \bar{\sigma}) = \theta.$$

Theorem 3. *If the real function* $\|\exp(At)\|$ *is bounded for* $t \geqslant 0$ *and the set* $\{\mathbf{b}, A\mathbf{b}, \ldots, A^m\mathbf{b}, \ldots\}$ *is a basis of the space* H*, then the dynamical system corresponding to equation* (1) *is stabilized.*

Proof. By transforming the Cauchy formula, we obtain

$$x(t) = \exp(At)\left(x_0 + \int_0^t [\exp(-A\tau)]\, b\, d\bar{\sigma}(\tau)\right). \tag{9}$$

It is necessary to establish the existence of such a control $\bar{\sigma}(t)$, that

$$\lim_{t\to\infty} \int_0^t [\exp(-A\tau)]\, b\, d\bar{\sigma}(\tau) = -x_0$$

or

$$\int_0^\infty [\exp(-A\tau)]\, b\, d\bar{\sigma}(\tau) = -x_0. \tag{10}$$

Applying term-by-term integration here as well, we obtain

$$-x_0 = b\int_0^\infty d\bar{\sigma}(\tau) - Ab\int_0^\infty \tau\, d\bar{\sigma}(\tau) + \cdots + (-1)^m A^m b \int_0^\infty \frac{\tau^m}{m!}\, d\bar{\sigma}(\tau) + \cdots \tag{11}$$

Let us take one of the expansions of the element $-x_0$ in the basis $\{\mathbf{b}, \mathbf{Ab}, \ldots, A^m b, \ldots\}$ (uniqueness of the expansion is not assumed)

$$-x_0 = \sum_{m=0}^{\infty} x_m^0 A^m b.$$

To ensure the fulfillment of condition (11), it is sufficient to choose the function $\bar{\sigma}(\tau)$ solving the moment problem

$$\int_0^\infty \tau^m d\bar{\sigma}(\tau) = (-1)^m x_m^0 m! \quad (m = 0, 1, 2, \ldots).$$

As is known ([3], p. 103, Theorem 3.11), this problem has an infinite set of solutions for arbitrary right-hand sides.

Let $\bar{\sigma}(t)$ be one such solution. Then

$$\lim_{t\to\infty}\left(x_0 + \int_0^t [\exp(-A\tau)]\, b\, d\bar{\sigma}(\tau)\right) = \theta. \tag{12}$$

Figure 5: Figure 5