Analysis of a certain class of systems of differential-difference equations by the method of separation of motions

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Abstract

An approximate method for analyzing systems of equations describing controllers with digital computers is presented. The method consists of the artificial introduction of a "small" parameter for a subset of derivatives or differences, which allows for reducing the order of the equations under consideration and utilizing the phase plane apparatus. It is shown that a necessary condition for the separation of m fast coordinates is the presence of m controlled coordinates in the system. A method for the asymptotic representation of the solution is provided, its convergence is proven for cases where the control satisfies Lipschitz conditions, and an example of a discrete relay system of the 3-th order is analyzed. 2 illustrations. 2 bibliographical references.

Full Text

Preamble

This work, published in 1967, addresses the control of linear discrete-time systems. We consider a system of the form:

$$\dot{x} = Ax + bu_k$$

where x is an n-dimensional state vector, b is a constant vector, and u_k is a control signal that remains constant over the sampling interval $kT_0 \leq t < (k+1)T_0$. Here, T_0 denotes the sampling period. Following the methodology established in [?], the continuous system (1) can be represented as a discrete-time difference equation:

$$x_{k+1} = Bx_k + du_k$$

where $x_k = x(kT_0)$, $B = \exp(AT_0)$, and $d = (\exp(AT_0)) \int_0^{T_0} [\exp(-As)] b \, ds$. The objective is to analyze the properties of such systems and develop control laws $u_k = \Phi[x(kT_0)]$ that ensure stability and desired performance characteristics.

2. Canonical Transformations and Controllability

To simplify the analysis of the discrete system (3), we employ linear transformations. Let $y = T_1 x$ be a transformation that maps the system into a canonical form. According to the criteria established in [?], the system is controllable if the vectors $d, Bd, \dots, B^{n-1}d$ are linearly independent. If the system is not fully controllable, it can be decomposed into a controllable subsystem of dimension $m < n$.

Specifically, we consider the case where the control enters through a reduced-order manifold. Let us define a transformation T_2 such that the system (4) is partitioned. For $m = 2$, the vectors d and Bd define the controllable subspace. We assume that the components $d_n \neq 0$ and proceed with a recursive transformation of the state variables. The resulting system equations involve coefficients b_{ij} and d_i derived from the matrices B and d.

The transformation process involves defining $q = (B - I)d$, where I is the identity matrix. If $q_i \neq 0$ for $i \leq n-1$, we can further reduce the system to a form suitable for applying specific control laws, such as relay or sliding mode control.

3. Asymptotic Analysis and Small Sampling Periods

When the sampling period T_0 is small, we can utilize asymptotic expansions. Let $v = T_0$ be a small parameter. The matrix B can be expanded as $B = \exp(Av) = E + Av + O(v^2)$. As $v \to 0$, the discrete system (3) approaches its continuous counterpart. We introduce scaled variables z and v to capture the behavior of the system across different time scales.

The transformed system can be written as:

$$\begin{aligned} z_{k+1} &= Pz_k + pv_{1k} \\ v_{1,k+1} &= Qz_k + R_0 v_k + d_n^0 u_k \end{aligned}$$

where P, Q, R_0 are matrices of appropriate dimensions and d_n^0 is the transformed control gain. Under the assumption that $v \to 0$, we can analyze the stability of the equilibrium point. We define a Lyapunov-like function to prove that the trajectories of the discrete system remain close to the trajectories of the continuous system (1). Specifically, for a given number of steps N, the difference between the discrete state z_k and the continuous state $z(t)$ is of the order $O(v^s)$, where $s > 0$.

The control law $u_k = \text{sign}(\sigma_k)$ is often employed in these systems. We demonstrate that for sufficiently small v, the system enters a neighborhood of the switching manifold and exhibits behavior analogous to a sliding mode in continuous systems. The stability conditions involve the spectral radius of the matrix $(P + pR_0)$ and the properties of the gain d_n^0.

4. Numerical Example and Simulation

To illustrate the theoretical results, we consider a third-order system. Let the state variables be y_1, y_2, y_3. Applying the transformation $z = T_2 y$, we obtain a system where the control u_k directly influences the higher-order derivatives. For a sampling period $T_0 = v$, the control law is defined as $u_k = \text{sign}(z_{1k} + \alpha z_{2k} + \beta z_{3k})$.

Simulation results indicate that the state trajectories z_1, z_2 converge to the origin. The phase plane analysis shows the existence of regions S_1, S_2, S_3 where the control signal remains constant. As v decreases, the "chattering" effect near the switching surface is reduced, and the discrete-time system accurately tracks the ideal sliding manifold of the continuous system.

Conclusion

This paper has presented a method for the analysis and synthesis of control laws for discrete-time linear systems. By utilizing canonical transformations and asymptotic expansions for small sampling periods, we have established conditions for stability and convergence. The results are applicable to the design of digital controllers for physical systems where the sampling rate is high relative to the system dynamics.

References

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Figures

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DIFFERENTIAL EQUATIONS

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ON CONTROLLABILITY IN HILBERT SPACE

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Let H be a separable Hilbert space; A be a linear bcounded operator mapping H into H, orobpawaming H into H, $b \in H$.

Consider the equation

$$\frac{dx}{dt} = Ax + bu, \qquad (1)$$

where $x \in H$, $u(t)$ is a scalar function of time t $(0 \leqslant t < \infty)$, called the control, them. Let us assume at first that $u(t) \in L_2$.

The function $x = x(t)$ (t is a real argument, $x(t) \in H$) will be called a solution of equation (1) for $u = u(t)$ in the interval $[0, T]$, if the guest equaelty holds almost everywhere in this interval

$$\frac{dx(t)}{dt} = Ax(t) + bu(t).$$

It is not difficult to check that the solution $x(t)$ of equation (1) wit ch initialy condition $x(0) = x_0$ (Cauchy formula)

$$x(t) = [\exp At]\, x_0 + \int_0^t [\exp A(t-\tau)]\, bu(\tau)\, d\tau.$$

We shall by definition call the genelalized solution of equation (1) the fynction

$$x(t) = [\exp At]\, x_0 + \int_0^t [\exp A(t-\tau)]\, bd\sigma(\tau), \qquad (2)$$

where $d\sigma(\tau)$ is the Stieltjes differential of a fynction of organed variation $\sigma(t)$ and, in particular, icllu if $\sigma(t)$ is a differentiable fynction, $d\sigma(t) = u(t)\,dt$.

The dynamical system described by equation (1) will be called controllable, if for ars many point $x_0 \in H$ there exists a fynction $\sigma(t)$ of organoed variation (admisssible control) and a finenue-utite time $t = T$ $(T > 0)$ such that $x(T) = \theta$. Here $x(t)$ is defined by formula (2), and θ denotes the origin of the unique prostpance H.

It is known that in the case when the space H is n–dimensional, the linear nesabecemoots of the vectors $b, Ab, A^2b, \ldots, A^{n-1}b$ obescnures the controllability of the system.

The quesson naturally arises, what role does the sequence being urpaes $b, Ab, A^2b, \ldots, A^nb, \ldots$, if the prostpance H is infinte-dimensional.

We shall call the system $\{e_m\}_0^\infty$ a basis of the space H, il if every element $x \in H$ can be represented (not necessarilly uniquely) in the form

$$x = \sum_{m=0}^{\infty} \alpha_m e_m,$$

where α_m — are numbers.

Figure 1: Figure 1

Theorem 1. *If system* (1) *is controllable, then the set*

$$E = \{b, Ab, A^2b, \ldots, A^m b, \ldots\}$$

is a basis in the space H.

Proof. Let x_0 be an arbitrary point of the space H. For this point, there exists an admissible control $\sigma(t)$ and a number T such that

$$x(T) = [\exp AT]\, x_0 + \int_0^T [\exp A(T-\tau)]\, b\, d\sigma(\tau) = 0.$$

From this we get

$$\int_0^T [\exp(-A\tau)]\, b\, d\sigma(\tau) = -x_0. \tag{3}$$

The integrand is represented as a series. We will show the possibility of term-by-term integration of this series.

Using Theorem 3. 7. 5 and Corollary 1 p. IV (see [1], pp. 95 and 78), we get

$$\left\| \int_0^T [\exp(-A\tau)]\, b\, d\sigma(\tau) - \sum_{m=0}^{n} \int_0^T (-1)^m \frac{\tau^m}{m!} A^m b\, d\sigma(\tau) \right\| =$$

$$= \left\| \int_0^T \left(\sum_{m=n+1}^{\infty} (-1)^m \frac{\tau^m}{m!} A^m b \right) d\sigma(\tau) \right\| \leqslant$$

$$\leqslant \left(\sup_{0 \leqslant \tau \leqslant T} \left\| \sum_{m=n+1}^{\infty} (-1)^m \frac{\tau^m}{m!} A^m b \right\| \right) \operatorname{Var}[\sigma(\tau)] \leqslant$$

$$\leqslant \left(\sup_{0 \leqslant \tau \leqslant T} \sum_{m=n+1}^{\infty} \frac{(\tau \|A\|)^m}{m!} \|b\| \right) \operatorname{Var}[\sigma(\tau)].$$

As $n \to \infty$, the last expression tends to zero, which proves the possibribility of term-by-term integration. Performing term-by-term integration in the left part of parity (3), we given

$$-x_0 = b \int_0^T d\sigma(\tau) - Ab \int_0^T \tau\, d\sigma(\tau) + \cdots + (-1)^m A^m b \int_0^T \frac{\tau^m}{m!}\, d\sigma(\tau) + \cdots \tag{4}$$

The theorem is proved.

It is easy to establish, that the infinite dimensionality of the linear span of the set E implies the linear independence of any finite number of elements of this set. Not a finite basis.

Now let the set E be a basis of various basicom space H in, more and, moreover, the following requirement (condition C) is for an arbitrary $x \in H$ the decomposition

$$x = \sum_{m=0}^{\infty} a_m A^m b$$

is determined uniquetely.

Figure 2: Figure 2

Condition C will, for example, be satisfied if in the coordinate Hilbert space ℓ_2 we take as b the vector $\{1, 0, 0, \ldots, 0, \ldots\}$, and set the operator A using the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & . & . & . \\ 1 & 0 & 0 & 0 & . & . & . \\ 0 & 1 & 0 & 0 & . & . & . \\ 0 & 0 & 1 & 0 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}.$$

Obviously, $Ab = \{0, 1, 0, \ldots\}$, $A^2b = \{0, 0, 1, 0, \ldots\}$ etc.

Condition C is highly restrictive. When it is satisfied, the operator A, for example, cannot have eigenvalues other than zero.

Indeed, let $Ax = \lambda x$, $\lambda \neq 0$, $x \neq \theta$. For simplicity, let us assume that λ is a real number. We represent the eigenvector x of the number λ in the form

$$x = \sum_{m=0}^{\infty} a_m A^m b.$$

By virtue of the continuity of operator A, we have

$$Ax = \sum_{m=0}^{\infty} a_m A^{m+1} b.$$

Using the equality $\lambda x = Ax$, we get

$$\sum_{m=0}^{\infty} a_m \lambda A^m b = \sum_{m=0}^{\infty} a_m A^{m+1} b.$$

Hence $a_0\lambda = 0$, therefore, $a_0 = 0$. From the same equality it follows further that $a_1 = 0$, $a_2 = 0$ and in general $a_m = 0$ for $m = 0, 1, 2, \ldots$ Thus, $x = \theta$ contrary to the assumption.

We shall now show that in an infinite-dimensional space H, Theorem 1 is not invertible, if condition C is satisfied.

Let some point $x_0 \in H$ be transferred by the control $\sigma(t)$ in time T to the initial point θ. Then equaelity (3) is satisfied, and, consequently, equanlity (4).

Let us assume that

$$x_0 = \sum_{m=0}^{\infty} x_{0m}^{m} A^m b.$$

Using equaality (4), we obtain

$$\int_0^T \tau^m d\sigma(\tau) = (-1)^{m+1} m!\, x_{0m}^{o} \quad (m = 0, 1, 2, \ldots). \tag{5}$$

Conversely, if there exist a number T and a function $\sigma(t)$ of organined variation, such that equanities (5) hold, then the point

$$x_0 = \sum_{m=0}^{\infty} x_{0m}^{o} A^m b$$

is transferred by the control $\sigma(t)$ in time T to the point θ.

Figure 3: Figure 3

Thus, the question of existence for a given point x_0 of a control that transfers this point in finite time to the position θ, is equivalent to the question of solvability of the power moment problem (5).

Let's make a change of variable, setting $\tau = Tt$, and denote $g(t) = \sigma(Tt)$. From equality (5) we obtain

$$\int_0^1 t^m\, dg(t) = (-1)^{m+1} x_0^m \frac{m!}{T^m} \qquad (m = 0, 1, 2, \ldots). \tag{6}$$

We use the following Hausdorff theorem ([2], p. 209).

For the existence of a function of bounded variation $g(t)$, such that

$$\int_0^1 t^m\, dg(t) = \mu_m \ \ (m = 0, 1, 2, \ldots),$$

it is necessary and sufficient that

$$\sum_{k=0}^{m} C_m^k |\Delta^{m-k} \mu_k| \leqslant M \ \ (m = 0, 1, 2, \ldots), \tag{7}$$

where C_m^k are binomial coefficients; M is some positive constant, and $\Delta^n \mu_k$ denote the n-th differences for the sequence $\{\mu_k\}$.

Theorem 2. If condition C is satisfied and, starting from some number $N > 0$, all numbers x_0^m in the expansion

$$x_0 = \sum_{m=0}^{\infty} x_0^m A^m b$$

are non-negative (or non-positive), and $x_N \neq 0$, then the point x_0 $(x_0 \neq \theta)$ cannot be transferred by any admissible control $\sigma(t)$ in finite time to the position θ (i.e. the point x_0 does not belong, as they say, to the domain of reachability of the initial point θ).

Proof. Indeed, let $x_0^m \geqslant 0$ for $m \geqslant N > 0$ and $x_0{}^N > 0$. Let

$$\mu_m = (-1)^{m+1} x_0^m \frac{m!}{T^m}. \tag{8}$$

for an arbitrary but fixed number $T > 0$.

The formula holds $(k < m)$

$$\Delta^{m-k} \mu_k = \mu_k - C_{m-k_1}^m \mu_{k+1} + C_{m-k_2}^m \mu_{k+2} + \cdots + (-1)^{m-k} \mu_m.$$

For $k = N$, using the condition of the theorem and formula (8), we obtain

$$|\Delta^{m-N} \mu_N| \geqslant |\mu_N| > 0.$$

Therefore, as $m \to \infty$, we have

$$\sum_{k=0}^{m} C_m^k |\Delta^{m-k} \mu_k| \to \infty.$$

Condition (7), necessary for the solvability of the moment problem, is not satisfied.

Theorem proved.

Figure 4: Figure 4

In particular, the points x_0, in the expansion of which only a finite number of coefficients x_m^0 differ from zero (among these numbers at least one has a positive index), do not belong to the reachable set of point 0.

A dynamical system described by equation (1), is called stabilizable, if for any point $x_0 \in \mathcal{H}$ there exists such an admissible control $\sigma(t)$, that the corrersonding solution $x(t, x_0, \sigma)$ of equation (1) possesses the property:

$$\lim_{t \to \infty} x(t, x_0, \sigma) = 0.$$

Theorem 3. If the essential function $\| \exp(At) \|$ is bounded for $t \geqslant 0$ and the set $\{b, Ab, A^2b, \dots, A^m b, \dots\}$ is a basis of the space H, then the dynamical system corresponding to equation (1), is stabilizable.

Proof. Transforming the Cauchy formula, we obtain

$$x(t) = \exp(At) \left(x_0 + \int_0^t [\exp(-A\tau)]\, b\, d\sigma(\tau) \right). \tag{9}$$

It is necessary to establish the existence of such a control $\sigma(t)$, that

$$\lim_{t \to \infty} \int_0^t [\exp(-A\tau)]\, b\, d\sigma(\tau) = -x_0$$

urd

$$\int_0^\infty [\exp(-A\tau)]\, b\, d\sigma(\tau) = -x_0. \tag{10}$$

Applying term-by-term integration here as well, we obtain

$$-x_0 = b \int_0^\infty d\sigma(\tau) - Ab \int_0^\infty \tau\, d\sigma(\tau) + \cdots + (-1)^m A^m b \int_0^\infty \frac{\tau^m}{m!}\, d\sigma(\tau) + \cdots \tag{11}$$

Let us take one of the expansions of the element $-x_0$ in the basis $\{b, Ab, \dots, A^m b, \dots\}$ (the uniqueness of the expansion is not assumed)

$$-x_0 = \sum_{m=0}^{\infty} x_m^0 A^m b.$$

In order to ensure the fulfollment of condition (11), it is suffictiont to select a function $\sigma(\tau)$, selving the moment problem

$$\int_0^\infty \tau^m\, d\sigma(\tau) = (-1)^m x_m^0\, m! \quad (m = 0, 1, 2, \dots).$$

As is insectn ([3], p. 103, theorem 3.11), this problem has an enconive mower set of solutions for anoitsary right-hand sides.

Let $\sigma(t)$ be one us takinx solutions. Then

$$\lim_{t \to \infty} \left(x_0 + \int_0^t [\exp(-A\tau)]\, b\, d\sigma(\tau) \right) = 0. \tag{12}$$

Figure 5: Figure 5

If the function $u(z_k, v_k)$ satistifies the Lipschitz conditions

$$|u(z', v') - u(z', v')| \leqslant L_1 \|z' - z'\| + L_2 \|v' - v'\|,$$
$$\left(\|z\| = \max_{i \in \overline{1, n-m}} |z_i|, \quad \|v\| = \max_{i \in \overline{1, m}} |v_i|\right),$$

then the convergence of successive approximations (17) is easily proved.

Theorem. For an arbierary natural number N, it is possible to indicate sach a number v_0, depending on P, ρ_0, R_0, d_0^0, L_1, u, L_2, that the sequence $\{z_k^{(i)}, v_k^{(i)}\}$ converges in horm to the solution of system (15) as $i \to \infty$ for all $k < N$ and $v \leqslant v_0$, wherein $\|z_k - z_k^{(i)}\| = O(v^{\varepsilon(i-1)})$ as $v \to 0$, $\varepsilon > 0$.

Droof. Denate by

$$\Delta_{k+1}^{(i)} z = z_k^{(i)} - z_k^{(i-1)}, \quad \Delta_k^{(i)} v = v_k^{(i)} - v_k^{(i-1)}.$$

From (17) it follows that

$$\|\Delta_{k+1}^{(i)} z\| \leqslant \alpha\nu \|\Delta_k^{(i)} z\| + \nu\beta \|\Delta_k^{(i-1)} v\|,$$
$$\|\Delta_{k-1}^{(i)} z\| \leqslant \gamma \|\Delta_k^{(i)} z\| + 6 \|\Delta_k^{(i)} v\|, \tag{18}$$

where

$$\alpha = \max_{s \in \overline{1, n-m}} \left\{\sum_{j=1}^{n-m} |p_{ij}^0|\right\}, \quad (p_{si}^0 = p_{si} - \delta_{si}),$$
$$\beta = \max_{s \in \overline{1, n-m}} |p_s^0|, \quad \gamma = \max_{j \in \overline{1, m}} \left\{\sum_{i=1}^{n-m} |q_{ji}^{(i)}| + L_1 |d_j^i|\right\},$$
$$\delta = \max_{j \in \overline{1, m}} \left\{\sum_{i=1}^{n-1} |r_{js}^0| + L_2 |d_j^0|\right\},$$
$$d_j^0 = 0, \quad j \in \overline{1, m-1}, \quad d_n^0 = d_n^0 \neq 0.$$

Substituting into the right side of the first of the inequelities (18) this inequarntity itself for $\|\Delta_k^{(i)} z\|$, then for $\|\Delta_{k-1}^{(i)} z\|$ z.., we will get (taking into account that $\Delta_0^{(i)}(\cdot) = 0$)

$$\|\Delta_{k+1}^{(i)} z\| \leqslant \nu\beta \sum_{s=1}^{k} \nu a^{k-s} \|\Delta_k^{(i-1)} z\|. \tag{19}$$

Similarlly, using $(k-1)$ times the second of the ineqaeities (18), we will get

$$\|\Delta_{k+1}^{(i)} z\| \leqslant \gamma \sum_{s=1}^{k} \delta^{k-s} \|\Delta_k^{(i)} z\|. \tag{20}$$

Substituting (20) into (19) and vice versa, we will get an estinate of the $i-1)$-th devieiation of $(i-1)$-th

$$\|\Delta_{k+1}^{(i)} z\| \leqslant \nu\beta\gamma \sum_{s=1}^{k} (\nu a)^{k-s} \sum_{l=1}^{s-1} \delta^{s-i-1} \|\Delta_k^{(i-1)} z\|,$$
$$\|\Delta_{k+1}^{(i)} v\| \leqslant \nu\beta\gamma \sum_{s=1}^{k} \delta^{k-s} \sum_{l=1}^{s-1} (\nu a)^{s-i-1} \|\Delta_l^{(i-1)} v\|. \tag{21}$$

Let N be an arbitrary natural number and

$$Z_i = \max_{s \in N} \|\Delta_k^{(i)} z\|, \quad V_i = \max_{s \in N} \|\Delta_k^{(i)} v\|.$$

Figure 6: Figure 6

Then from (21) it follows

$$Z_i \leqslant \nu f Z_{i-1} \quad \left(f = \frac{\beta\gamma}{\delta - 1}\left[\delta\,\frac{\delta^N - (\nu\alpha)^N}{\delta - \nu\alpha} - \frac{1 - (\nu\alpha)^N}{1 - \nu\alpha}\right]\right),$$
$$V_i \leqslant \nu\varphi\, V_{i-1} \quad \left(\varphi = \frac{\beta\gamma}{\nu\alpha - 1}\left[\frac{\delta^N - (\nu\alpha)^N}{\delta - \nu\alpha} - \frac{\delta^N - 1}{\delta - 1}\right]\right). \tag{22}$$

Obviously, for νf, $\nu\varphi$, smaller than 1 (by the way, f and φ are always > 0), the sequences for $\{Z_i\}$, $\{V_i\}$ monotonically converge to zero. Moreover, for a fixed N, the quantities f and φ are finite, and if $f \leqslant f_0 \nu^{-(1-\varepsilon)}$, then $Z_i = O(\nu^{\varepsilon}) Z_{i-1}$. Similarly, if for sucficiently small vulue ν $\varphi \leqslant \varphi_0 \nu^{-(1-\varepsilon)}$, then also

$$V_i \leqslant \varphi_0 \nu^{\varepsilon} V_{i-1} \leqslant \ldots \leqslant \nu^{i\varepsilon} \varphi_0^i V_0.$$

The theorem is proved.

Note that mat for convergence for any N, it is sufficient that the quantity δ be less than 1. (In this case vmrue $\varepsilon = 1$).

The proved theorem establishes the fact that the successive approxim-approximations (17) asymptotically converge to the solution of system (14) (oor (4)). The obtained convergence conditions are rather rigid, and mainy inapplicable for the case of discontinuous control, and monzer caes on εmrm-prenmoud. More promising, in our opinion, is the solution of the problem of statnibility of the system (15) with respect to the first approximation, defined by formylas (16) or (17).

4. Example. As an illustration, let's consider the following system:

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = x_3, \quad \frac{dx_3}{dt} = -Ku_k(x). \tag{23}$$

Obviously,

$$B = \exp AT_0 = \begin{vmatrix} 1 & T_0 & \frac{1}{2}T_0^2 \\ 0 & 1 & T_0 \\ 0 & 0 & 1 \end{vmatrix}; \qquad d = -K \begin{vmatrix} \frac{T_0^3}{3!} \\ \frac{T_0^2}{2!} \\ T_0 \end{vmatrix},$$

and system (3) will take the form

$$x_{k+1} = Bx_k + du_k. \tag{24}$$

The transformation T_1 consists of the following:

$$y_1 = x_1 - \frac{T_0^2}{6}x_3, \quad y_2 = x_2 - \frac{T_0}{2}x_3, \quad y_3 = x_3, \tag{25}$$

and in coordinates y system (24) will be written as:

$$y_{k+1} = \begin{vmatrix} 1 & T_0 & T_0^2 \\ 0 & 1 & T_0 \\ 0 & 0 & 1 \end{vmatrix} y_k - K \begin{vmatrix} 0 \\ 0 \\ T_0 \end{vmatrix} u_k. \tag{26}$$

The transformation $T_2 (z = T_2 y)$ will be expressed by the formulas:

$$z_1 = y_1 - T_0 y_2, \quad z_2 = y_2, \quad z_3 = y_3, \tag{27}$$

Figure 7: Figure 7

and in coordinates z the system will take the form

$$z_{k+1} = \begin{bmatrix} 1 & T_0 & 0 \\ 0 & 1 & T_0 \\ 0 & 0 & 1 \end{bmatrix} z_k - K \begin{bmatrix} 0 \\ 0 \\ T_0 \end{bmatrix} u_k. \qquad (28)$$

Let $m = 2$, $K = v^{-2} K_0$, $T_0 = v$, $v_1 = z_2$, $v_2 = v z_3$. Then system (14) will become the system

$$\begin{aligned} z_{1,k+1} &= z_{1,k} + v v_{1,k}, \\ v_{1,k+1} &= v_{1,k} + v_{2,k}, \\ v_{2,k+1} &= v_{2,k} - K_0 u_k. \end{aligned} \qquad (29)$$

Note that $z = T_2 T_1 x$ and

$$T_2 T_1 = \begin{bmatrix} 1 & -T_0 & \frac{T_0^2}{3} \\ 0 & 1 & -\frac{T_0}{2} \\ 0 & 0 & 1 \end{bmatrix}; \quad T_1^{-1} T_2^{-1} = \begin{bmatrix} 1 & T_0 & \frac{1}{6} T_0^2 \\ 0 & 1 & \frac{1}{2} T_0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let us incestigate the case of relay control

$$u_k = \operatorname{sign} x_{1k}.$$

Then in coordinates z, v

$$u_k = \operatorname{sign} \sigma_k, \quad \sigma_k = z_{1k} + v v_{1,k} + \frac{v}{6} v_{2,k}.$$

The zero-approximation equations will take the form

$$\begin{aligned} z_k^{(0)} = z_{10}; \quad v_{2,k+1}^{(0)} &= v_{1,k}^{(0)} + v_{2,k}^{(0)}, \\ v_{2,k+1}^{(0)} &= v_{2,k}^{(0)} - K_0 \operatorname{sign} \sigma_k^{(0)}. \end{aligned} \qquad (30)$$

The trajectories of system (30) lie in the place $z_1 = z_{10}$ and pass throw the points of parabolas

$$\begin{aligned} 2K_0 v_1 + \left(v_2 + \frac{K_0}{2}\right)^2 &= \text{const for } \sigma > 0, \\ -2K_0 v_1 + \left(v_2 - \frac{K_0}{2}\right)^2 &= \text{const for } \sigma < 0. \end{aligned} \qquad (31)$$

Due to time quantization, switching can occur at any by source at anby point in sectors S_1 and S_2, enclosed respectively between the lines:

$$\begin{aligned} &S_1: \quad \sigma^{(0)} = 0; \quad z_1 + v v_1 - \frac{5}{6} v v_2 = -\frac{5}{6}, K_0; v_2 > 0; \\ &S_2: \quad \sigma^{(0)} = 0; \quad z_{10} + v v_1 - \frac{5}{6} v v_2 = \frac{5}{6}, K_0; v_2 < 0 \\ &\left(\sigma^{(0)} = z_1 + v v_1 + \frac{v}{6} v_2\right). \end{aligned} \qquad (32)$$

Figure 8: Figure 8

Considering point transformations of S_1 into S_2 and S_2 into S_1 in the system (30), it man be sokown, that many point of sector S_1 with coordinate v_{20} in onne obolotion around the point $v_2 = 0$, $v_1 = -z_{10}/v$ will have the coordinate $v_2'' > v_{20}$. Consequently, from a qavesttative standnint, the nero appfriximation first nyre danmation (dmstions in the (v_1, placorenno (v_1, v_2) plocrines) will representaant a cniprall unswiesling around the poimt $v_2 = 0$, $v_1 = -\frac{z_{10}}{v}$.

Tinkcy on cvegrem (during the time of odno obolution in the (v_1, v_2) plactine) the snak of v_1 snacs z_{10}, to opotuвonomint a snak on z_1 bydet yeninate in modylus, ho a moment appfrimination a byres кoopdunates v_1 u v_2 bydet ument coopdinate sncturs. Tho finnvedet ko fact, the snak кoopdinates z_1 ismenrates to npotuвonomoxiical u велеvuna $|z_{1k}|$ bydet vennocraes npurmily do credymmucs nepechowning. Due to the fact, that this first dmnration ane a computornnosn representanion are unswiing cnipaly, the repexident process for z_1 repressents divornging koleбation with an yeslication dlungi perioods.

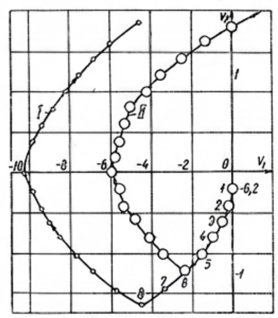

Fig. 1. Phase portrait of first domation: I — curve of nyro appfrimization; II — curve of first appfrimization (which is also experimemental, panculated on a TsVM), trajectory of the first domation

Thus, dawe passmotration of the nyro appfrimization aвsolves one to draw a suisod about the kavecthennal portugn of dbimation in custeme (23) (und (29)): the nyro pelietion of custems is unnetfinable.

In pic. 1, a phase portrait of first donation in the (v_1, , v_2) is presented, oblynned by method of sucreommessive appfriimizations on manually and pasculated on a TsVM, and in pic. 2, graph of demenges in x_{1k}, nolynned on a TsVM. Danneis pacvet fully nodtherzpidt the suisod about the kavecthennow xapractery dbimation.

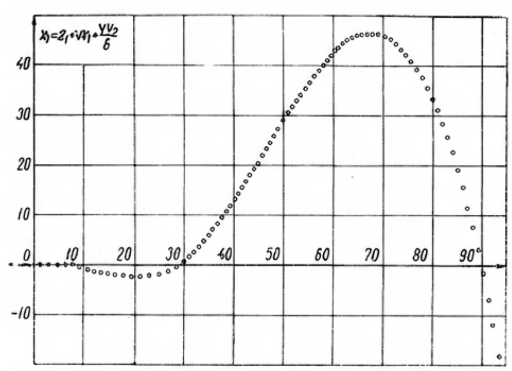

Fig. 2. Results of digital computer calculation of the transient process for x_1

Figure 9: Figure 9

Conclusions

1. The method of separation of motions for discrete control systems allows establishing the qualitative nature of motion with a nonlinear type of control action.

2. The type of control $u_b(x)$ does not play such a significant role as in known methods [1], due to the reduction of order and the possibility of using the phase plane apparatus.

3. With the method of separation of motions, the nonlinear system is considered as substantially nonlinear.

The author thanks Professor E. A. Barbashin for his attention to this work.

N o t e i n p r o o f. As shown by further studies and calculation of specific examples, provided by the author and V. M. Reshetov, the method outlined in the article gives satisfactory accuracy with a sufficiently large discretization interval — T_0 (more precisely, when $d_n^0 = d_n - v^{(m-1)} = O(1)$). If d_n^0 is small (for example, $d_n^0 = O(v)$), then it is advisable to first separate the motions in the system of differential equations (1), and then proceed to a recurrent system of type (3).

Literature

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Figure 10: Figure 10