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**Abstract**

**Full Text**

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*MATHEMATICS*

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## UNIFORM ASYMPTOTICS OF LINEAR SYSTEMS

*(Presented by Academician M. V. Keldysh, 17 V 1966)*

A new method is proposed for studying the asymptotics of systems of linear equations with a small parameter at the derivative

$$\varepsilon \frac{d\bar{x}}{dt} = A(t, \varepsilon)\bar{x}. \quad (1)$$

The fundamental matrix of the system is represented in the form

$$S(t, \varepsilon) = W(t, \varepsilon)\Lambda(t, \varepsilon), \quad (2)$$

where the matrix  $\Lambda$  is diagonal, while the matrix  $W(t, \varepsilon)$  has a limit as  $\varepsilon \rightarrow 0$ . The determination of the matrix  $W$  can be reduced to the computation of the factors of the infinite right product

$$W = W_0 W_1 \dots W_n \dots, \quad (3)$$

with each factor  $W_n$  found by means of quadratures, and the product converges faster than the numerical product

$$\prod_{n=0}^{\infty} (1 + C\varepsilon^{\Phi_n}). \quad (4)$$

Here  $\Phi_n$  are Fibonacci numbers, growing as  $\left(\frac{\sqrt{5}+1}{2}\right)^n$ . In the particular case of matrices of second order, the rate of convergence is still higher and attains the rate of convergence of Newton's method.

The validity of the assertions made is proved on any interval  $t_0 \leq t \leq t_1$  on which the real parts of the eigenvalues of the matrix  $A(t, \varepsilon)$  do not intersect.

**1. The matrix  $A$  is triangular.** Suppose, in addition, that the diagonal elements of the matrix  $A$ , which in this case coincide with its eigenvalues, are arranged in decreasing order of their real parts

$$\operatorname{Re} \lambda_1(t, \varepsilon) > \operatorname{Re} \lambda_2(t, \varepsilon) > \dots > \operatorname{Re} \lambda_k(t, \varepsilon). \quad (5)$$

In this important particular case the matrix  $W$  is also triangular, and its determination reduces to quadratures.

Let us write out the equation satisfied by the matrix  $S(t, \varepsilon)$ :

$$\varepsilon dS/dt = AS, \quad (6)$$

and, substituting (2) into (6), multiply by  $\Lambda^{-1}$  on the right and by  $W^{-1}$  on the left. We obtain

$$\varepsilon W^{-1} \dot{W} + \varepsilon \dot{\Lambda} \Lambda^{-1} = W^{-1} A W. \quad (7)$$

We shall seek a triangular matrix  $W$  with ones on the diagonal. Then the left-hand side of equality (7) splits into two nonintersecting matrices. The second term has nonzero elements only on the diagonal, and the first only off the diagonal. It is not difficult to show that in this case the single equation (7) uniquely determines two equations for  $W$  and  $\Lambda$ . Let us, however, first introduce notation convenient for what follows—

tion. Let  $z = \|z_{il}\|$  be an arbitrary matrix. Denote by  $T(z)$ ,  $\lambda(z)$ ,  $\perp(z)$  its lower triangular, diagonal, and upper triangular parts, respectively:

$$T(z) = \left( \left( \begin{array}{cccc} 0 & 0 & \dots & 0 \\ z_{21} & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ z_{n1} & z_{n2} & \dots & 0 \end{array} \right) \right), \quad \lambda(z) = \left( \left( \begin{array}{cccc} z_{11} & 0 & \dots & 0 \\ 0 & z_{22} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & z_{nn} \end{array} \right) \right),$$

$$\perp(z) = \left( \left( \begin{array}{cccc} 0 & z_{12} & \dots & z_{1n} \\ 0 & 0 & \dots & z_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 0 \end{array} \right) \right). \quad (8)$$

Let us now write down the equations for  $\Lambda$  and  $W$ :

$$\varepsilon \dot{\Lambda} = \lambda \Lambda, \quad (9)$$

$$\varepsilon \dot{W} = TW + \lambda W - W \lambda, \quad (10)$$

where  $\lambda = \lambda(A)$  and  $T = T(A)$ .

Equation (9) is integrated immediately:

$$\Lambda(t, \varepsilon) = \exp \left[ \frac{1}{\varepsilon} \int_{t_0}^t \lambda(\tau, \varepsilon) d\tau \right]. \quad (11)$$

Equation (10), by means of the intermediate substitution  $W = \Lambda U \Lambda^{-1}$ , can be written in the form of the integral equation

$$W(t) = E + \frac{1}{\varepsilon} \int_{t^*}^t \exp \left[ \frac{1}{\varepsilon} \int_{\tau}^t \lambda(s) ds \right] T(\tau) W(\tau) \exp \left[ -\frac{1}{\varepsilon} \int_{\tau}^t \lambda(s) ds \right] d\tau. \quad (12)$$

If this equation is iterated  $k$  times, an exact solution is obtained, since the product of more than  $k$  triangular matrices with zeros on the diagonal ( $k$  is the dimension of the matrices) is identically equal to zero. Moreover, by direct verification one can see that all the exponents standing under the integral sign will have negative real part. This follows (of course, after multiplying the exponential factors) again from the triangularity of the matrices  $T$ ,  $W$  and the ordering (see (5)) of the eigenvalues of the matrix  $A$ . Therefore one may put  $t^* = t_0$ , and the matrix  $W(t, \varepsilon)$ , determined by equation (12), will have a finite limit as  $\varepsilon \rightarrow 0$ . This limit is the solution of the equation

$$TW + \lambda W - W\lambda = 0, \quad (13)$$

which is obtained from (10) if one formally sets  $\varepsilon = 0$ . Equation (13) has a unique solution among triangular matrices  $W$  (with ones on the diagonal). From formula (12) one can derive the estimate

$$W = E + O(T),$$

valid for small  $T$ , which plays an important role in what follows. In the estimate, the denominators contain differences of the eigenvalues (their real parts) of the matrix  $A$ . Therefore condition (5) is essential for the validity of the estimate.

**Remark.** In a completely analogous way one can consider the case when the matrix  $A$  has zeros below the diagonal. It is only necessary everywhere to write  $\perp(A)$  instead of  $T(A)$  and, in formula (12), to put  $t^* = t_1$ .

**2. The matrix  $A$  is almost triangular.** We now consider a somewhat more general case, when the matrix  $A$  is not exactly triangular, but its deviation from triangularity is small. Suppose, for example, that the estimate holds:

$$\perp(A) \sim \delta,$$

where  $\delta$  is some small quantity.

Again we seek the solution of the equation for the fundamental matrix of the system in the form of a product of matrices:

$$S = P\Sigma, \quad \varepsilon P^{-1}\dot{P} + \varepsilon\dot{\Sigma}\Sigma^{-1} = P^{-1}AP.$$

An arbitrary choice arises in specifying the equation for  $P$ , and we use this arbitrariness by setting

$$\varepsilon\dot{P} = TP + \lambda P - P\lambda.$$

The equation for  $P$  has been written as though the matrix  $A$  were triangular. If it actually were triangular, then the equation for  $\Sigma$  would give us a diagonal matrix. But since the matrix  $A$  is only almost triangular, we expect that for  $\Sigma$  we shall obtain an almost diagonal equation. After this one may hope to improve the situation further by repeating the procedure. Let us write the equation for  $\Sigma$ :

$$\varepsilon\dot{\Sigma} = B\Sigma,$$

where

$$B = P^{-1} \perp(A)P + \lambda(A).$$

Our expectations are indeed justified, since now not only the “upper,” but also the “lower” parts of the matrix  $B$  turn out to be of order  $\delta$ . At the same time we see that the equation for  $\Sigma$  has the same form as the original equation for  $S$ , but with the changed matrix  $B$ . The matrix  $B$  is better than  $A$ , since although the upper triangular part has not decreased, the lower one has decreased. It is therefore natural to take the next step and correct the upper triangular part.

These considerations suggest the expediency of the following iterative process. We start with a certain matrix  $A_n$  and construct successively four matrices  $P_n, B_n, Q_n, A_{n+1}$ :

$P_n$  is the solution of the equation

$$\varepsilon\dot{P}_n = T(A_n)P_n + \lambda(A_n)P_n - P_n\lambda(A_n); \quad (14)$$

$B_n$  is given by the formula

$$B_n = P_n^{-1} \perp (A_n)P_n + \lambda(A_n); \quad (15)$$

$Q_n$  is again the solution of the equation

$$\varepsilon \dot{Q}_n = \perp (B_n)Q_n + \lambda(B_n)P_n - P_n\lambda(B_n) \quad (16)$$

and, finally,  $A_{n+1}$  is given by the formula

$$A_{n+1} = Q_n^{-1}T(B_n)Q_n + \lambda(B_n). \quad (17)$$

This method resembles the alternating method in conformal mappings. Here we likewise reduce first one half of the matrix, then the other. Note that the diagonal elements remain throughout of order unity and, in the principal terms, coincide with the initial ones. Let us see what we have achieved as a result of one such double step. The estimate is obtained by using the formula

$$P_n = E + O[T(A_n)].$$

Substituting into (15) and retaining only the principal terms, we find that

$$B_n \approx \lambda(A_n) + \perp (A_n) + O(T(A_n) \perp (A_n)).$$

Consequently,

$$\perp (B_n) \sim \perp (A_n), \quad T(B_n) \sim \perp (A_n)T(A_n).$$

In exactly the same way, on the next half-step we obtain

$$\perp (A_{n+1}) \sim \perp (B_n)T(B_n), \quad T(A_{n+1}) \sim T(B_n).$$

These relations ensure rather rapid convergence, though not as rapid as in Newton's method, since the error is squared not at every stage, but only after one stage. It is not difficult to find the rate at which the off-diagonal terms decrease. Let

$$T(A_n) \sim \delta^{p_n}, \quad \perp (A_n) \sim \delta^{q_n};$$

then from the formulas written above we find

$$T(B_n) \sim \delta^{p_n + q_n}, \quad \perp (B_n) \sim \delta^{q_n}$$

and further, for the second half of the step, we obtain

$$\mathbb{T}(A_{n+1}) \sim \delta^{p_n+q_n}, \quad \perp(A_{n+1}) \sim \delta^{p_n+2q_n}.$$

We can therefore write, by induction,

$$p_{n+1} = p_n + q_n, \quad q_{n+1} = p_n + 2q_n;$$

since for  $n = 0$  we had  $p_0 = 0$ ,  $q_0 = 1$ , the resulting sequence, which gives (as is easy to see) an upper estimate, coincides with the Fibonacci numbers. More accurate calculations show that, for matrices of second order, the rate of convergence obtained is exactly the same as in Newton's method. It is curious that two is the first convergent of the number  $(\sqrt{5} + 1)/2$ , which determines the asymptotics of the Fibonacci numbers. It is possible that more accurate calculations will give, for matrices of third order, a rate determined by the third (one must take approximations by excess) convergent; for matrices of fourth order—the fifth, and so on. However, more detailed estimates have not been carried out.

**3. An arbitrary differentiable matrix  $A(t, \varepsilon)$ .** In this case the problem is easily reduced to the problem of the preceding section. It is only necessary to make a nonstandard first step in the iterative process.

Indeed, let us consider the original equation for the matrix  $S$  and set, as usual,  $S = W_0 \Sigma$ . For  $\Sigma$ , as has already been checked three times, an equation of the same form as for  $S$  is obtained, but with the matrix  $B$

$$B = W_0^{-1} A W_0 - \varepsilon W_0^{-1} \dot{W}_0.$$

We now choose the matrix  $W_0$  so that the first, principal, term in the expression for  $B$  is triangular. There always exists, and indeed a unitary, matrix  $W_0$  which solves this problem. Its construction is very simple. One must take the system of eigenvectors of the matrix  $A$  and orthonormalize it. The matrix of transition from the coordinate unit vectors to the orthonormal basis thus obtained solves the problem of finding the matrix  $W_0$ . There are exactly  $k!$  ways of choosing such a matrix  $W_0$ . However, for our purposes only two are suitable. One arranges the eigenvalues in increasing order of their real parts, and the other arranges them in decreasing order.

It should be noted that the unitary matrix  $W_0(t, \varepsilon)$  thus obtained will have a bounded derivative only on an interval where there is no crossing of eigenvalues. At points where the matrix  $A(t)$  has multiple roots, the derivative  $\dot{W}_0$  generally becomes infinite. This is easily checked, for example, in the case of a turning point in the Schrödinger equation.

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*Note: Figure translations are in progress. See original paper for figures.*

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