ON OPEN FINITE-TO-ONE MAPPINGS

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MATHEMATICS

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ON OPEN FINITE-TO-ONE MAPPINGS *

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This work is closely connected with the recent works of A. V. Arkhangel' skii ([3]) and V. V. Proizvolov ([6]). Its main results are the following theorems.

Theorem 1. Let $f : X \to X$ be an open finite-to-one mapping of a regular p-space ** X onto a metrizable space Y. Then X is metrizable.

Theorem 2. The inverse image and the image of a hereditarily weakly paracompact space under open finite-to-one mappings are hereditarily weakly paracompact spaces (in the class of T_2-spaces).

Theorem 3. Let $f : X \to Y$ be an open finite-to-one mapping of a regular space X onto a Hausdorff hereditarily paracompact space Y. Then X too is hereditarily paracompact.

Remark. For the proof of Theorem 2 (3) it is sufficient to prove that the space X is weakly paracompact (paracompact), since every open subspace $U \subseteq X$ is mapped openly and finite-to-one onto fU.

We shall say that a set $A \subseteq X$ is weakly paracompact (paracompact) in X if into every cover of the set A open in X one can inscribe some point-finite (locally finite) cover in X of the set A by sets open in X.

Let $f : X \to Y$ be a finite-to-one mapping. By M_n we shall agree to denote the totality of all points of the space X of multiplicity n with respect to the mapping f (i.e. $M_n = \mathcal{E}\{x : f^{-1}fx$ consists of n points$\}$).

Lemma 1. Let $f : X \to Y$ be an open finite-to-one mapping of a Hausdorff space X onto a space Y. If the set fM_n is weakly paracompact in Y, then the set M_n is weakly paracompact in X.

Proof. Let $\omega = \{U_\alpha\}$ be some cover, open in X, of the set M_n. Since the mapping f is open and finite-to-one, for any n points $x_1, x_2, \dots, x_n \in M_n$, for which $fx_1 = fx_2 = \dots = fx_n$, one can construct such open sets $Vx_1, Vx_2, \dots, Vx_n$ that $Vx_i \cap Vx_j = \emptyset$ for $i \neq j$, $fVx_1 = fVx_2 = \dots = fVx_n$, and each set Vx_i $(i = 1, 2, \dots, n)$ is contained in some set $U_\alpha \in \omega$. Denote by $\eta = \{Vx\}$ the totality of the sets constructed by us, which, by construction, is inscribed in ω

and covers the set M_n. Into the cover $f\eta = \{fVx\}$, open in Y, of the set fM_n, inscribe an open point-finite in Y cover $\xi = \{G_\beta \mid \beta \in \theta\}$ of the set fM_n.

The system of sets open in X, $f^{-1}\xi = \{f^{-1}G_\beta \mid \beta \in \theta\}$, is point-finite and covers M_n. Each set G_β is contained in some fVx; hence there exist such sets $Vx_1, Vx_2, \dots, Vx_n \in \eta$ that $fVx_1 = fVx_2 = \dots = fVx_n = fVx$. Put $\Gamma_{\beta i} = f^{-1}G_\beta \cap Vx_i$ and prove

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* The mapping $f : X \to Y$ is finite-to-one if the inverse image of every point consists of a finite number of points. Let us note that all mappings are assumed to be continuous and onto.

** The class of p-spaces was introduced and studied in the paper ([4]).

the following relation:

$$\bigcup_{i=1}^{n} \Gamma_{\beta i} \cap M_n = f^{-1}G_\beta \cap M_n \tag{a}$$

It is enough to verify that

$$\bigcup_{i=1}^{n} \Gamma_{\beta i} \cap M_n \supset f^{-1}G_\beta \cap M_n,$$

and the latter follows from the obvious relation

$$\bigcup_{i=1}^{n} Vx_i \cap M_n = f^{-1}fVx \cap M_n.$$

For each set G_β we fix some $fVx \supset G_\beta$ and put $\gamma = \{\Gamma_{\beta i} \mid \beta \in \theta;\ i = 1, 2, \dots, n\}$. On the basis of formula (a), the system γ covers the set M_n, and, by construction, γ is inscribed in ω. Let now x be an arbitrary point of the space X. By the hypothesis, the point x is contained in no more than finitely many elements of the system $f^{-1}\xi$–say, in $f^{-1}G_{\beta_1}, f^{-1}G_{\beta_2}, \dots, f^{-1}G_{\beta_k}$. Then x can belong only to the sets $\Gamma_{\beta_l i}$, where $l = 1, \dots, k$ and $i = 1, 2, \dots, n$. Therefore the system γ is point-finite in X. Lemma 1 is proved.

Similarly one proves

Lemma 2. *Let $f : X \to Y$ be an open finite-to-one mapping of a Hausdorff space X onto a space Y. If the set fM_n is paracompact in the subspace*

$$Y_n = Y \setminus \bigcup_{i=1}^{n-1} fM_i \quad (\text{where } Y_1 = Y),$$

then the set M_n is paracompact in the subspace

$$X_n = X \setminus \bigcup_{i=1}^{n-1} M_i \quad (\text{where } X_1 = X).$$

Proof of Theorem 2. If the space X is hereditarily weakly paracompact, then it is not hard to see that the space Y is also hereditarily weakly paracompact. Let now the space Y be hereditarily weakly paracompact and let $\omega = \{U_\alpha \mid \alpha \in \theta\}$ be some open cover of the space X.

From Theorem 1 of [2] it follows that, for any natural number n, the set

$$\bigcup_{i=1}^{n} M_i$$

is closed in X. In view of the hereditary weak paracompactness of the space Y, the sets M_n and fM_n, for any natural number n, satisfy Lemma 1; consequently, one can find an open system ω_n, point-finite in X, which covers the set M_n and is inscribed in the system

$$\omega_n' = \left\{ U_\alpha \cap \left(\bigcup_{j=n}^{\infty} M_j \right) \mid \alpha \in \Theta \right\}.$$

It is clear that the cover

$$\omega' = \bigcup_{n=1}^{\infty} \omega_n$$

of the space X is point-finite. This proves Theorem 2.

Theorem 4. *The preimage and the image of a hereditarily finally compact space under open finite-to-one mappings is again a hereditarily finally compact space (in the class of Hausdorff spaces).*

The proof of this fact is analogous to the proof of Theorem 2; only, instead of point-finite systems, one must consider countable systems.

Proof of Theorem 3. Let Y be a hereditarily paracompact space. By Lemma 2, the set M_1 is paracompact in X. Suppose that we have already proved that the set

$$\bigcup_{i=1}^{n} M_i$$

is paracompact in X. Consider an arbitrary cover ω of the set

$$\bigcup_{i=1}^{n+1} M_i.$$

By the hypothesis, in ω one can inscribe some open locally finite cover

$$\omega_1 = \{U_\alpha \mid \alpha \in A\}$$

of the set

$$\bigcup_{i=1}^{n} M_i,$$

and, by Lemma 2, also some open ⋯

a locally finite in $X \setminus \bigcup_{i=1}^{n} M_i$ cover $\omega_2 = \{V_\beta \mid \beta \in B\}$ of the set M_{n+1}. Since X is regular and, by assumption, the set $\bigcup_{i=1}^{n} M_i$ is paracompact in X, there exists an open set $\Gamma \supset M_{n+1} \setminus G$ in X, where $G = \bigcup_{\alpha \in A} U_\alpha$, such that

$$[\Gamma] \cap \left(\bigcup_{i=1}^{n} M_i\right) = \emptyset.$$

Therefore the system $\omega_2' = \{V_\beta \cap \Gamma \mid \beta \in B\}$ is locally finite in X, and, consequently, the cover $\omega' = \omega_1 \cup \omega_2'$ is locally finite in X.

Thus, the set $\bigcup_{i=1}^{n} M_i$ is paracompact in X for every natural number n; consequently, it can be separated from any closed set in X that does not meet it. Let now γ be an arbitrary open cover of the space X. For every natural number n, by Lemma 2, there exists a certain σ-discrete in

$$X_n = X \setminus \bigcup_{i=1}^{n-1} M_i \quad (X_1 = X)$$

cover

$$\gamma_n = \{U_{\alpha n} \mid \alpha \in A\}$$

of the set M_n, inscribed in γ. Suppose that σ-discrete in X systems

$$\gamma_1' = \{\Gamma_{\alpha 1} \mid \alpha \in A\}, \ldots, \gamma_n' = \{\Gamma_{\alpha n} \mid \alpha \in A\},$$

have been constructed, covering the set $\bigcup_{i=1}^{n} M_i$, and moreover $\gamma_1' = \gamma_1$. The system γ_{n+1}', σ-discrete in X, is constructed as follows: put

$$G = \bigcup_{i=1}^{n} \left(\bigcup_{\alpha \in A} \Gamma_{\alpha i}\right)$$

and take such an open in X_n set Γ that

$$\Gamma \supset M_{n+1} \setminus G \quad \text{and} \quad [\Gamma] \cap \left(\bigcup_{i=1}^{n} M_i \right) = \emptyset.$$

The system

$$\gamma'_{n+1} = \{ \Gamma_{\alpha(n+1)} = \Gamma \cap U_{\alpha(n+1)} \mid \alpha \in A \}$$

is the desired one.

Thus, into the cover γ we have inscribed the σ-discrete cover

$$\gamma' = \bigcup_{n=1}^{\infty} \gamma'_n.$$

On the basis of Proposition 1 from (5), the space X is paracompact.

Proposition 1. *If a Hausdorff topological space X is mapped openly and finite-to-one onto a perfectly normal paracompact space Y, then into every open cover ω of the space X one can inscribe an open σ-discrete cover γ.*

The proof of this fact is analogous to the proof of Theorem 3. Theorem 1 follows from Theorem 3 and from a theorem of A. V. Arhangel' skii from (3).

Example 1. The points of the space R are all points of the segment $[0, 1]$. At all points $x \neq 0$, the neighborhoods are the same as on the half-interval $(0, 1]$ (see (1), p. 863). As neighborhoods of the point $x = 0$ take sets of the form $[0, \varepsilon) \setminus D$, where $D = \{1, 1/2, ..., 1/n, ...\}$ and $0 < \varepsilon < 1$. The space R is not paracompact (it is not regular), although it is finally compact. Take $I = [0, 1]$ in the usual topology. Put $Z = R \cup I$, where $[R]_Z \cap [I]_Z = \emptyset$. Glue the points $0 \in R$ and $0 \in I$ into one; endow the resulting set with the quotient topology and denote it by X. The space X: 1) is Hausdorff; 2) is not paracompact; 3) is mapped openly and finite-to-one onto the segment $I = [0, 1]$; 4) is a p-space (as the sum of a countable number of bicompacts of countable character in X). This example shows that Theorems 1 and 3 cannot be extended from regular spaces to Hausdorff ones.

Example 2. Let X be the well-known Niemytzki space. It is not difficult to verify that X has a countable refining sequence of covers; hence it is a p-space. Denote by $L_0 \subseteq X$ the set of real numbers, and by $L_n \subseteq X$ a line parallel-

the line L_0, which is at distance $1/n$ from L_0. Put

$$Z = \bigcup_{i=0}^{\infty} L_i.$$

The space Z is a subspace of the space X. The space Z: 1) is completely regular; 2) is a p-space; 3) is not weakly paracompact; 4) is mapped openly, countably, and compactly onto the set of real numbers L_0 in the usual topology (it is enough to map points of the form $(x, 0)$, $(x, 1/n)$ to the point x). This example shows that the preceding theorems cannot be extended to countable-to-one mappings. Example 1 from (6) shows that the word "hereditarily" in Theorems 2, 3, and 4 cannot be omitted. There exists an example (see (6), Example 2) of a finite-to-one open mapping of a perfectly normal space with the first axiom of countability onto a compact space; therefore the requirement of "featheredness" in the formulation of Theorem 1 is essential.

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Note: Figure translations are in progress. See original paper for figures.