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**Abstract**

**Full Text**

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## THE SPECTRAL RADIUS OF INDECOMPOSABLE POSITIVE OPERATORS

*(Presented by Academician S. L. Sobolev on 22 XII 1966)*

In this note we shall use the notation and definitions introduced in ([1]).

It is known ([2]) that if $T \in [Y]$ is a semi-non-supporting operator and if $T$ has property $(S)$, then the point $\lambda = \rho(T)$ belongs to the spectrum $\sigma(T)$, and there correspond to it eigen-elements $x_0 \in K,\ x'_0 \in K'$ such that $\langle x, x'_0 \rangle > 0$ for all nonzero vectors $x \in K$, and $\langle x_0, x' \rangle > 0$ for all nonzero functionals $x' \in K'$. In other words, the semi-non-supporting operator is indecomposable in the sense of the definition of Stetsenko ([3]). It is also known that the converse assertion holds ([3]).

Everywhere in what follows it is assumed that $K$ is a reproducing and normal cone ([4]).

**Theorem 1.** *Suppose that:*

1. *$T \in [Y]$ is a positive operator such that its spectral radius $\rho(T)$ is an eigenvalue. Let this eigenvalue have corresponding to it an eigenvector $x_0 \in K$.*
2. *$x \in K,\ x' \in K'$ are elements such that there exist $\alpha > 0$ and a positive integer $q$ such that $T^q x > \alpha x_0$ and $\langle x_0, x' \rangle > 0$.*

*Then*

$$\rho(T) = \lim_{p\to\infty} \left[\langle T^p x, x' \rangle\right]^{1/p}.$$

The validity of this formula is obvious in the case $\rho(T) = 0$, and in the case $\rho(T) > 0$ follows from the relations

$$0 < \rho(T)\alpha^{1/p}[\rho(T)]^{-q/p}\left[\langle x_0, x' \rangle\right]^{1/p} \le \left[\langle T^p x, x' \rangle\right]^{1/p} \le$$

$$\le \rho(T)\left\|\frac{1}{[\rho(T)]^p}T^p\right\|^{1/p}\|x\|^{1/p}\|x'\|^{1/p}.$$

**Remark.** Condition 1 of Theorem 1 is fulfilled if:

(a) The operator $T \in [Y]$ has property $(S)$ and $TK \subset K$.

Condition 2 is fulfilled if (a) holds and

(b) $T$ is semi-non-supporting and $u_0$ is an operator bounded below ([5]), where $u_0 \in K,\ \|u_0\| = 1$, and $(u_0 - xx_0) \in K,\ x_0 = [\rho(T)]^{-1}Tx_0,\ x_0 \in K,\ x_0 \neq 0$.

Then the only restrictions on $x \in K,\ x' \in K'$ are the conditions $x \neq 0,\ x' \neq 0$.

**Theorem 2.** *Suppose that:*

($\alpha$) $T \in [Y]$ *is a positive operator.*

($\beta$) $\hat{x}$ *is a non-supporting element of the cone* $K$*.*

($\gamma$) $H'$ *is a* $K$*-total set such that* $\langle \hat{x}, x' \rangle = 1$ *for all* $x' \in H'$*.*

*If we put*

$$r(T) = \inf_{x' \in H'} \langle T\hat{x}, x' \rangle, \qquad R(T) = \sup_{x' \in H'} \langle T\hat{x}, x' \rangle,$$

then the inequalities

$$r(T) \leq [r(T^2)]^{1/2} \leq \cdots \leq [r(T^{2^p})]^{2^{-p}} \leq \cdots \leq \rho(T) \leq \cdots$$
$$\cdots [R(T^{2^p})]^{2^{-p}} \leq \cdots \leq [R(T^2)]^{1/2} \leq R(T).$$

Suppose, in addition:

($\delta$) $T$ has property $(S)$.

($\varepsilon$) $T$ is a semi-nonsupporting operator.

($\eta$) There exist positive numbers $\alpha,\ \beta$ and a positive integer $q$

such that $\alpha x_0 < T^q x < \beta x_0$, where $x_0 = [\rho(T)]^{-1}Tx_0,\ x_0 \in K,\ x_0 \neq 0$. Then the equalities hold

$$\rho(T) = \lim_{p\to\infty} [r(T^{2^p})]^{2^{-p}} = \lim_{p\to\infty} [R(T^{2^p})]^{2^{-p}}.$$

For the proof of Theorem 2 we introduce functionals, defined on $K \setminus \{0\}$,

$$r_x(T) = \inf_{\substack{x' \in H' \\ \langle x, x' \rangle \neq 0}} \frac{\langle Tx, x' \rangle}{\langle x, x' \rangle}, \qquad r^x(T) = \sup_{x' \in H'} \frac{\langle Tx, x' \rangle}{\langle x, x' \rangle}.$$

An obvious consequence of the $K$-totality of the set $H'$ is the inequalities

$$r_x(T) \le r_{Tx}(T) \le \dots \le r_{T^q x}(T) \le \dots \le r^{T^q x}(T) \le \dots \le r^{Tx}(T) \le r^x(T).$$

Using these inequalities and the hypotheses of Theorem 2, we obtain the relations

$$0 \le r_{T^{2^p}\hat{x}}(T^{2^p}) - r_{\hat{x}}(T^{2^p}) = \inf_{x' \in H'} \frac{\langle T^{2^p+1}\hat{x}, x'\rangle}{\langle T^{2^p}\hat{x}, x'\rangle} - \inf_{x' \in H'} \langle T^{2^p}\hat{x}, x'\rangle,$$

$$0 \le r^x(T^{2^p}) - r^{T^{2^p}\hat{x}}(T^{2^p}) = \sup_{x' \in H'} \langle T^{2^p}\hat{x}, x'\rangle - \sup_{x' \in H'} \frac{\langle T^{2^p+1}\hat{x}, x'\rangle}{\langle T^{2^p}\hat{x}, x'\rangle}$$

and, after simple calculations, the desired result.

The class of operators that we consider is quite broad. Let us give two typical examples. Some classes of indecomposable operators are considered in ([3]).

**Example 1.** Let $Y$ be an $m$-dimensional Banach space. Then, in a certain basis of the space $Y$, the semi-nonsupporting operator $T$ is represented by an indecomposable matrix $(t_{jk})$, where $t_{jk} \ge 0,\ 1 \le j, k \le m$. If $\hat{x}(t, \dots, 1)^*$, where the asterisk denotes that $x$ is a column vector,

$$x'_j = (0, \dots, \underbrace{1}_{j}, 0, \dots, 0),$$

$$H' = \{x'_j \mid j = 1, \dots, m\}, \qquad r_j(T) = \sum_{k=1}^{m} t_{jk},$$

$$r(T) = \min_{j=1,\dots,m} r_j(T), \qquad R(T) = \max_{j=1,\dots,m} r_j(T),$$

then we obtain the theorems to which Yamamoto refers in ([6]).

**Example 2.** Let $Y = C(\langle 0, 1\rangle)$, and let $T$ be an integral operator with kernel $\tau = \tau(s, t) \ge 0$, continuous on $\langle 0, 1\rangle \times \langle 0, 1\rangle$. Let $\hat{x} = \hat{x}(s) \equiv 1$,

$$H' = \{\delta_s \mid 0 \le s \le 1,\ \delta_s = \delta(s) \text{ is the Dirac delta function}\}.$$

Then

$$r(T) = \min_{s \in \langle 0,1\rangle} \int_0^1 \tau(s, t)\, dt \le \rho(T) \le \max_{s \in \langle 0,1\rangle} \int_0^1 \tau(s, t)\, dt = R(T).$$

If, moreover, the kernel $\tau$ is such that for an arbitrary nonnegative function $y \in C(\langle 0,1\rangle)$, $y(s) \neq 0$, there is a positive integer $q = q(y)$ such that

$$\int_0^1 \int_0^1 \cdots \int_0^1 \tau(s, t_q) \cdots \tau(t_1, t) y(t)\, dt\, dt_1 \cdots dt_q > 0,$$

then

$$\lim_{p\to\infty} \{[R(T^{2^p})]^{2^{-p}} - [r(T^{2^p})]^{2^{-p}}\} = 0.$$

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6. T. Yamamoto, Numer. Math., 8, No. 4 (1966).

*Note: Figure translations are in progress. See original paper for figures.*