Application of the asymptotic method to the integration of autonomous systems with lag

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Abstract

A quasilinear autonomous system with many degrees of freedom and delays is considered. Under the assumption that the generating system possesses a multi-frequency periodic regime, the solution of the system is constructed using the asymptotic method of N. M. Krylov and N. N. Bogolyubov. The possibility of the existence of periodic solutions in the system is investigated, and a stability criterion is established, which coincides in form with the one obtained by the small parameter method. Bibliography 4.

Full Text

Preamble

This work builds upon the foundational methods established in 1967 by R. E. Kushner [2] and N. M. Krylov and N. N. Bogolyubov [2]. We consider the systems discussed in [1], focusing on the stability and periodic solutions of differential-difference equations. Building on the results in [3], we examine a system of equations of the form:

$$x(t) = \{x_1(t), \dots, x_n(t)\}$$

where the dynamics are governed by functions $F = \{F_1, \dots, F_n\}$ and G. We assume the existence of time delays $0 < \tau_1 < \tau_2 < \cdots < \tau_p$. The functions F_n are assumed to be bounded such that $|F_n| < \mu$, where μ is a small parameter.

We analyze the characteristic equation associated with the linearized system:

$$(E - e^{a}) \det(1.2) = 0$$

Let N_j denote the roots of the characteristic equation. The solution to the homogeneous system can be expressed in terms of $x(t) = \pm N_j$. For the perturbed system, we consider the operator:

$$AGx(t - \tau_0) \tag{1.3}$$

The solution to (1.3) is sought in the form of a series expansion (1.4). We define the fundamental frequency and the corresponding eigenfunctions $\phi_k = \{\phi_{1k}, \dots, \phi_{nk}\}$ as described in [4], associated with the eigenvalues $\pm N_j$. For the case where $\tau = 0$, the eigenfunctions take the form $\phi_{s1} = B_s$, where B_s is a constant. More generally, the components are given by:

$$\phi_{sj}(\theta) = C_{sj} \cos N_j\theta + D_{sj} \sin N_j\theta$$

where the coefficients C_{sj} and D_{sj} are determined by the boundary conditions of (1.2) and (1.3).

2. Method of Construction

Following the approach in [2], we represent the solution to (1.1) in the following form:

$$x(t) \approx V + \mu U_1(\theta, M_1, \dots, M_{m-1}) + \dots \tag{2.1}$$

where $U_i = [U_{i1}, \dots, U_{in}]$ and M_k are functions to be determined. We assume the parameters M_k satisfy the following system of equations:

$$\frac{dM_k}{dt} = \mu A_k + \mu^2 B_k + \dots \tag{2.2}$$

Substituting (2.1) and (2.2) into the original system (1.1), and collecting terms of the same order of μ, we obtain a sequence of equations for U_i:

$$(L - A)U_i = f_i \tag{2.3}$$

where L is the linear operator corresponding to the unperturbed system. The functions $F_0, F_1, \dots$ represent the expansion of the nonlinear part $F[x(t-\tau), \dots, x(t)]$. Using the substitution $\Phi_k = \phi_k[\theta - \alpha(c_0 + \mu \dots)]$, we can express the derivatives and the delayed terms. The resulting equations for the first and subsequent approximations are:

$$\sum \Phi_k(\theta - \tau_j\omega) - \dots = F_0 \tag{2.5}$$

$$\sum \Phi_k(\theta - \tau_j) + \alpha M_k(\theta) = R_i \tag{2.6}$$

where R_i depends on the previous approximations U_{m-1} and M_{m-1}. The solvability conditions for these equations, as established in [4], require that the right-hand sides be orthogonal to the solutions of the adjoint system $\psi_j = \{\psi_{1j}, \dots, \psi_{nj}\}$. The adjoint system is defined by:

$$\frac{dy(t)}{dt} = A' G y(t + \tau_0)$$

where A' is the transpose of the matrix A. Following the procedure in [4], we determine the constants a_{kj} such that the resonance conditions are satisfied. Specifically, we require $a_{kj} = \frac{1}{2\pi}\delta_{kj}$, leading to the following conditions for M_k:

$$P_j(M_1, \dots, M_{m-1}) = 0 \tag{2.9}$$

The solution for U_i can then be written as:

$$U_i = \Phi_k \Phi_k^*(\theta) + f_i(\theta, M_1, \dots, M_{m-1}) \tag{2.10}$$

3. Stability and Convergence

We now consider the stability of the periodic solutions for (1.1). Let $M_k = M_k(\mu)$ be the solutions to the averaged equations (2.2). At $\mu = 0$, we denote these values as $M_k(0) = M_k'$. The equilibrium points M_k' must satisfy the stationary condition (2.9):

$$P_j(M_1', \dots, M_{m-1}') = 0 \tag{3.2}$$

The perturbed values are then expressed as:

$$M_k = M_k' + \mu\rho_k + \dots \tag{3.3}$$

As shown in [4], if the Jacobian of the system (3.2) is non-zero at the equilibrium point, then for sufficiently small μ, there exists a unique periodic solution to (1.1) that corresponds to the steady-state solution (2.1). The stability of this solution is determined by the eigenvalues of the matrix $\frac{\partial P_k}{\partial M_j}$.

4. Conclusion

We have demonstrated that the periodic solutions of the differential-difference system can be constructed using an asymptotic expansion in terms of the small parameter μ. By introducing the corrected frequencies $M_k = M_k' + \Delta M_k$, we ensure that the resonance terms are properly accounted for in the higher-order approximations. The conditions for the existence and stability of these solutions are given by the properties of the averaged equations (4.5). This approach extends the classical results of Krylov-Bogolyubov to systems with multiple time delays.

References

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Note: Figure translations are in progress. See original paper for figures.