ON THE SPATIAL CAUCHY-POISSON PROBLEM ON WAVES ON THE SURFACE OF A VISCOUS LIQUID OF FINITE DEPTH

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Abstract

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ON THE SPATIAL CAUCHY-POISSON PROBLEM ON WAVES ON THE SURFACE OF A VISCOUS LIQUID OF FINITE DEPTH

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We shall assume that the liquid, extending without bound in the horizontal direction, is bounded above by a free surface and below by a fixed horizontal plane. We take the surface of the liquid in the equilibrium position as the coordinate plane $z = 0$ and direct the axis Oz vertically upward. We denote the depth of the liquid by h. We shall suppose that at the initial instant of time the velocity of the liquid particles is zero and the shape of the free surface is prescribed. On the free surface the pressure $p = p_0(x, y, t)$ is prescribed and the tangential stress is absent. We regard the motion of the liquid as slow, the wave amplitude as small, and the waves as long. Then in the Navier-Stokes equations the nonlinear terms may be discarded, and the problem reduces to integrating, in dimensionless variables, the equations

$$\begin{gathered} \partial v_x/\partial t = -\partial p/\partial x + \Delta v_x, \qquad \partial v_y/\partial t = -\partial p/\partial y + \Delta v_y, \\ \partial v_z/\partial t = -\partial p/\partial z + \Delta v_z, \qquad \partial v_x/\partial x + \partial v_y/\partial y + \partial v_z/\partial z = 0, \\ p = p' + \lambda^3 z, \qquad \lambda^3 = gh^3\nu^{-2} \end{gathered} \tag{1}$$

with the boundary and initial conditions

$$\begin{gathered} \partial v_x/\partial z + \partial v_z/\partial x = 0, \qquad \partial v_y/\partial z + \partial v_z/\partial y = 0, \\ -p + \lambda^3\zeta + 2\partial v_z/\partial z = -p_0(x, y, t), \qquad \partial\zeta/\partial t = v_z \quad \text{for } z = 0; \end{gathered} \tag{2}$$

$$v_x = v_y = v_z = 0 \quad \text{for } z = -1; \tag{3}$$

$$v_x = v_y = v_z = 0, \qquad \zeta = \zeta_0(x, y) \quad \text{for } t = 0. \tag{4}$$

Here p' is the dimensionless hydrodynamic pressure.

The dimensional coordinates x_1, y_1, z_1, velocity $\mathbf{v}_1$, time t_1, dynamic part of the pressure p_1, and elevation of the free surface ζ_1 are expressed in terms of the dimensionless variables as follows:

$$x_1 = hx, \qquad y_1 = hy, \qquad z_1 = hz,$$

$$\mathbf{v}_1 = \frac{\nu}{h}\mathbf{v}, \qquad t_1 = \frac{h^2}{\nu}t, \qquad \zeta_1 = h\zeta, \qquad p_1 = \rho\frac{\nu^2}{h^2}p. \tag{5}$$

Let us apply to equations (1), to the boundary and initial conditions (2)-(4), the multiple Fourier transform ([2]) with respect to the variables x and y, and then the Laplace transform ([2]) with respect to the time t, taking the initial conditions into account. Solving the resulting boundary-value problem for the system of ordinary differential equations, we find, after applying the inversion formulas for the Laplace and Fourier transforms and the convolution theorem ([2]) for the Laplace transform,

$$\zeta = \zeta_0 - \frac{1}{2\pi}\int_0^t \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_0(\xi,\eta,t-\tau)\Psi(a,\tau)e^{-i(\xi x+\eta y)}\,d\xi\,d\eta\,d\tau -$$

$$-\frac{\lambda^3}{2\pi}\iint\limits_{-\infty}^{\infty} H_0(\xi,\eta)\Phi(a,t)e^{-i(\xi x+\eta y)}\,d\xi d\eta; \tag{6}$$

$$\Psi(a,\tau) = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\frac{af_1(b,a)}{f_2(b,a)}e^{s\tau}ds,$$

$$\Phi(a,t) = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\frac{af_1(b,a)}{f_2(b,a)}e^{st}\frac{ds}{s}; \tag{7}$$

$$H_0(\xi,\eta) = \frac{1}{2\pi}\iint\limits_{-\infty}^{\infty}\zeta_0(x,y)e^{i(\xi x+\eta y)}\,dx\,dy,$$

$$P_0(\xi,\eta,t) = \frac{1}{2\pi}\iint\limits_{-\infty}^{\infty}p_0(x,y,t)e^{i(\xi x+\eta y)}\,dx\,dy; \tag{8}$$

$$
\begin{aligned}
f_1(b,a) &= b\ \mathrm{sh}\, a\ \mathrm{ch}\, b - a\ \mathrm{ch}\, a\ \mathrm{sh}\, b,\\
f_2(b,a) &= \lambda^3 a f_1(b,a) + (a^2+b^2)^2 f_3(b,a) -\\
&\quad - 4a^3 b\,(b\ \mathrm{sh}\, a\ \mathrm{sh}\, b - a\ \mathrm{ch}\, a\ \mathrm{ch}\, b) - 4a^2 b(a^2+b^2),\\
f_3(b,a) &= b\ \mathrm{ch}\, a\ \mathrm{ch}\, b - a\ \mathrm{sh}\, a\ \mathrm{sh}\, b,\\
a^2 &= \xi^2+\eta^2, \qquad b^2 = a^2 + s,
\end{aligned} \tag{9}
$$

s is the parameter of the Laplace transform. Formulas (6)–(9) give the exact solution of the problem posed.

To compute the integrals (7), we use the identity

$$
\frac{a f_1(b,a)}{f_2(b,a)} = \frac{a\ \mathrm{th}\, a}{(a^2+b^2)^2 + \lambda^3 a\ \mathrm{th}\, a} - \frac{a f_1(b,a)}{f_2(b,a)} \frac{E(a,b)\,\mathrm{th}\, a}{(a^2+b^2)^2+\lambda^3 a\ \mathrm{th}\, a},
$$

$$
E(a,b) = \frac{(a^2+b^2)^2 a\ \mathrm{sh}\, b}{\mathrm{sh}\, a\cdot f_1(b,a)} + \frac{4a^4 b\ \mathrm{ch}\, a}{\mathrm{ch}\, b\cdot f_1(b,a)} - \frac{4a^2 b(a^2+b^2)}{f_1(b,a)} - 4a^3 b\ \mathrm{th}\, b. \tag{10}
$$

Applying to (10) the inversion formula (7) and the convolution theorem, we shall have

$$
\Psi(a,t) = \Psi_0(a,t) - \int_0^t \Psi(a,\tau) K(a,t-\tau)\,d\tau,
$$

$$
K(a,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{E(a,b)\,\mathrm{th}\, a}{(a^2+b^2)^2+\lambda^3 a\ \mathrm{th}\, a} e^{st}\frac{ds}{s}. \tag{11}
$$

An analogous equation also holds for the function $\Phi(a,t)$.

We regard λ as large. Accurate up to terms containing multipliers λ^{-3}, we obtain

$$
\Phi(a,t) = \frac{1}{\lambda^3}\left[1 - e^{-2a^2 t}\cos\sqrt{\lambda^3 a\ \mathrm{th}\, a}\, t\right]\left[1 + O\!\left(\frac{1}{\lambda^{3/2}}\right)\right]; \tag{12}
$$

$$
\Psi(a,t) = \sqrt{\frac{a\ \mathrm{th}\, a}{\lambda^3}}\, e^{-2a^2 t}\sin\sqrt{\lambda^3 a\ \mathrm{th}\, a}\, t\left[1 + O\!\left(\frac{1}{\lambda^{3/2}}\right)\right]. \tag{13}
$$

Substituting (12) and (13) into (6). With the aid of the change of variables

$$
\xi = \lambda r\cos\theta, \qquad \eta = \lambda r\sin\theta, \qquad x = \lambda R\cos\vartheta, \qquad y = \lambda R\sin\vartheta \tag{14}
$$

we transform (6) into the form

$$\zeta = \frac{\lambda^2}{2\pi}\left\{\int_0^\infty\int_0^{2\pi} H_0(r,\theta)e^{-2\lambda^2 r^2 t}\cos\left(\lambda^2\sqrt{r\ \mathrm{th}\,\lambda r}\,t\right)e^{-i\lambda^2 Rr\cos(\theta-\vartheta)}r\,dr\,d\theta - \right.$$

$$-\frac{1}{\lambda}\int_0^t\int_0^\infty\int_0^{2\pi} P_0(r,\theta,t-\tau)\,r\sqrt{r\ \mathrm{th}\,\lambda r}\,e^{-2\lambda^2 r^2 t}\times$$

$$\left.\times\sin\left(\lambda^2\sqrt{r\ \mathrm{th}\,\lambda r}\,t\right)e^{-i\lambda^2 Rr\cos(\theta-\vartheta)}\,d\tau\,dr\,d\theta\right\}\left[1+O\left(\lambda^{-3/2}\right)\right]. \tag{15}$$

The evaluation of the integrals (15) is possible only for a specific choice of the functions $\zeta_0(x,y)$ and $p_0(x,y,t)$. If $H_0(r,\theta)$ and $P_0(r,\theta)$ are not oscillating functions, then, in order to evaluate the integrals (15), we apply successively twice the method of stationary phase ([1]). The large parameter here is λ^2. The results of the calculations for large values of $gt^2/4R$ may be written, returning to dimensional variables:

$$\zeta = \left\{\int_0^t \frac{g^2\tau^2}{8\sqrt{2R^3}}\exp\left[-2\nu g^2\tau^5/16R^4\right]\sqrt{\mathrm{th}\left(\frac{g\tau^2}{4R}\frac{h}{R}\right)}\times\right.$$

$$\times\left[\frac{1}{2i}\exp\left[i\frac{g\tau^2}{4R}\,\mathrm{th}\left(\frac{g\tau^2}{4R}\frac{h}{R}\right)\right]P_0\left(\frac{g\tau^2}{4R},\vartheta,t-\tau\right)+\right.$$

$$\left.\left.+\frac{1}{2i}\exp\left[-i\frac{g\tau^2}{4R}\,\mathrm{th}\left(\frac{g\tau^2}{4R}\frac{h}{R}\right)\right]P_0\left(\frac{g\tau^2}{4R},\vartheta+\Pi,t-\tau\right)\right]d\tau\right\}$$

$$-\frac{g^2t^2}{8\sqrt{2R^3}}\exp\left[-2\nu\frac{g^2t^5}{16R^4}\right]\sqrt{\mathrm{th}\left(\frac{gt^2}{4R}\frac{h}{R}\right)}\left[\frac{1}{2}H_0\left(\frac{gt^2}{4R},\vartheta\right)\times\right.$$

$$\left.\times\exp\left[i\frac{gt^2}{4R}\,\mathrm{th}\left(\frac{gt^2}{4R}\frac{h}{R}\right)\right]+\frac{1}{2}H_0\left(\frac{gt^2}{4R},\vartheta+\Pi\right)\exp\left[-i\frac{gt^2}{4R}\,\mathrm{th}\left(\frac{gt^2}{4R}\frac{h}{R}\right)\right]\right]\times$$

$$\times\left[1+O\left(\frac{1}{\lambda^{3/2}}\right)+O\left(\frac{4R}{gt^2}\right)\right]. \tag{16}$$

Formula (16) gives the solution of the posed problem for the case of large λ and $gt^2/4R$.

Consider examples.

1. Let

$$\zeta_0 \equiv 0, \qquad p_0 = A\delta(x)\delta(y)\delta(t). \tag{17}$$

Then from (16) it follows that

$$\zeta = -A\frac{gt^3}{8\sqrt{2\rho R^4}} \exp\left[-2\nu\frac{g^2t^5}{16R^4}\right] \sin\left[\frac{gt^2}{4R}\,\mathrm{th}\left(\frac{gt^2}{4R}\frac{h}{R}\right)\right] \times$$
$$\times\sqrt{\mathrm{th}\left(\frac{gt^2}{4R}\frac{h}{R}\right)}\left[1 + O\left(\frac{1}{\lambda^{3/2}}\right) + O\left(\frac{4R}{gt^2}\right)\right]. \tag{18}$$

Formula (18) makes it possible to solve the problem of ship waves.

2. Let

$$p_0 \equiv 0, \qquad \zeta_0 = \frac{A}{2\pi r}\delta(r). \tag{19}$$

Then from (15) it follows that

$$\zeta = A\frac{g^2t^2}{2^{7/2}R^3} \exp\left[-2\nu\frac{g^2t^5}{16R^4}\right] \cos\left[\frac{gt^2}{4R}\,\mathrm{th}\left(\frac{gt^2}{4R}\frac{h}{R}\right)\right] \times$$
$$\times\sqrt{\mathrm{th}\left(\frac{gt^2}{4R}\frac{h}{R}\right)}\left[1 + O\left(\frac{1}{\lambda^{3/2}}\right) + O\left(\frac{4R}{gt^2}\right)\right]. \tag{20}$$

As $h \to \infty$, (18) and (20) yield the known results ([3]).

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[3] A. K. Nikitin, S. A. Podrezov, PMM, 28, no. 3 (1964).

Note: Figure translations are in progress. See original paper for figures.