FINITE GROUPS CLOSE TO SPLIT GROUPS

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A finite group in which the condition (QP) is satisfied: *the intersection of any two distinct nonprimary maximal nilpotent subgroups of even order is the identity subgroup*, will be called a **QP-group**, and a nonprimary maximal nilpotent subgroup of even order a μ_2**-subgroup**. The condition (QP) is hereditary for subgroups and factor groups, as follows from the properties given below and from the centralizer criterion for QP-groups. Throughout, only finite groups are considered.

Lemma 1. Let G be a group of even order with nontrivial center $Z(G)$. Then, in order that G be a QP-group, it is necessary and sufficient that G be a group of one of the following types:

1. A nilpotent group.
2. $G = A\lambda T$, where T is a Sylow 2-subgroup of G, and T has a G-invariant subgroup $T_0 \neq E$ such that G/T_0 is a Frobenius group with invariant set isomorphic to A.
3. $G = T\lambda G_1$, where T is a Sylow 2-subgroup of G, and G_1 contains a nilpotent G-invariant subgroup G_0 such that G/G_0 is a Frobenius group with invariant set isomorphic to T.

Theorem 1. A group is a QP-group if and only if the centralizer of each of its nonidentity elements has at most one μ_2-subgroup.

Corollary. A group is a QP-group if and only if the centralizer of each of its nonidentity elements either has odd order or is a group of one of types 1–3 of Lemma 1.

The condition (QP) is clearly satisfied if the group has only one μ_2-subgroup. It turns out that such groups are solvable and are exhausted by the following types:

1–3. As in Lemma 1.

4. $G = S\lambda A\lambda\{b\}$, where S and $\{b\}$ are 2-groups, $S\lambda A$ is of type 3, and $A\lambda\{b\}$ is a Frobenius group with invariant set A.

Lemma 2. Let in a QP-group G one of the following conditions be satisfied:

(a) in G there are nonprimary elements of even order and a G-invariant 2-subgroup distinct from E;

(b) in G there are elements of order $2p$ (p an odd prime number) and a G-invariant p-subgroup distinct from E.

Then in G there exists only one μ_2-subgroup.

Corollary. The solvable radical of a nonsolvable QP-group is trivial, unless G is a CJT-group.

A **CJT-group** is a group in which the centralizer of any involution is a 2-group ([1]). It is clear that every CJT-group is a QP-group. Suppose that G is a solvable QP-, but not a CJT-group. Let the order of G be even. If G has a G-invariant 2-subgroup distinct from E, then by Lemma 2 G has exactly one μ_2-subgroup; consequently, G is a group of one of types 1–4. Suppose that in G there is more than one-

μ_2-subgroup. Then G has an invariant 2-complement, since its Sylow 2-subgroup is either cyclic or the quaternion group. Hence, applying induction on the order of the group G, we obtain the following theorem.

Theorem 2. *In order that a solvable group G of even order be a QP-group, it is necessary and sufficient that G be a group of one of the following types:*

1–4. As above.

5. A Frobenius group whose complement is a group of even order of one of types 1, 2.

Next we consider nonsolvable QP-groups. Let G be a nonsolvable QP-, but not a CJT-group. Then in G there exist μ_2-subgroups, and all of them are distinct from their normalizers, by the well-known theorem of Frobenius. Let H be one of these normalizers. By Theorem 2, H is a group of one of types 2–4. Suppose that H is a group of type 3. Then a Sylow 2-subgroup of H is also a Sylow 2-subgroup in G. From condition (QP) it follows that the Sylow 2-subgroups in G are pairwise mutually prime. From the description of such groups (2) it follows that G must be isomorphic to $SL(2,q)$ or $Sz(q)$, where q is a power of 2. However, the latter groups are CJT-groups. Thus, H is a group of one of types 2, 4.

Introduce the following notation: $\mathfrak{M}_1$ is the collection of normalizers of μ_2-subgroups of G, each of which is of type 4; $\mathfrak{M}_2$ is the collection of normalizers of μ_2-subgroups of G, each of which is a Hall subgroup in G of type 2; $\mathfrak{M}_3$ is the collection of normalizers of μ_2-subgroups of G, each of which is a group of type 2, but is not a Hall subgroup in G.

Consider three cases.

1st case: $\mathfrak{M}_1 \neq \emptyset$. If $K = S\lambda A\lambda\{b\} \in \mathfrak{M}_1$, then $T = S\{b\}$ coincides with its normalizer in G, and therefore is a Sylow 2-subgroup in G, and all subgroups from $\mathfrak{M}_1$ are conjugate. The normalizer $N(A)$ of the subgroup A in G is a Frobenius group $N(A) = A\lambda B$, where the complement B is a subgroup of the centralizer of a certain involution. Hence it follows that K is a Hall subgroup in G. If in the group G $\mathfrak{M}_2 \cup \mathfrak{M}_3 \neq \emptyset$, and U is the Fitting radical of K, then $K/U \approx S_3$, and the order of the Sylow 2-subgroup of the Fitting radical of any subgroup X from $\mathfrak{M}_2 \cup \mathfrak{M}_3$ is equal to 2. Using this assertion, one can prove that if $N(A) \neq A\lambda\{b\}$, then in $\mathfrak{M}_1$ there is a subgroup K_1 such that $K \cap K_1 = A\lambda\{b\}$. The same conclusion is reached by the assumption that there exists in $\mathfrak{M}_1$ a subgroup K_2, distinct from K and such that $K \cap K_2$ contains elements of odd order. All subgroups of G whose order is equal to the order of A are conjugate. Comparing the number of these subgroups in G, in K, and the cardinality of the set $\mathfrak{M}_1$ shows that if $N(A) = A\lambda\{b\}$, then the intersection of any two distinct subgroups from $\mathfrak{M}_1$ is a 2-group. Suppose that the intersection of any two distinct subgroups from $\mathfrak{M}_1$ is a 2-group, and denote by D a maximal one of these intersections. D is a maximal intersection of Sylow 2-subgroups of G. Therefore $N(D)$ is not 2-closed. If D is a cyclic group, then $N(D) \in \mathfrak{M}_2 \cup \mathfrak{M}_3$, and, as shown above, $|D| = 2$. If D is noncyclic, then D is an elementary abelian group of rank 2, $N(D)$ is a CJT-group of type 4, and $N(D)/D \approx S_3$. After these preliminary considerations one can prove the following lemma.

Lemma 3. *In a nonsolvable QP-, but not CJT-group G, a Sylow 2-subgroup is a dihedral group if $\mathfrak{M}_1 \neq \emptyset$.*

Proof. Suppose the contrary. From the definition of groups of type 4 and from the structure of 2-groups containing a self-centralizing subgroup of order 4, it follows that in G there are no self-centralizing subgroups of order 4. Suppose that $\mathfrak{M}_2 \cup \mathfrak{M}_3 \neq \emptyset$. If in $\mathfrak{M}_1$ there were two distinct subgroups whose intersection is not a 2-group, then the order of the subgroup A would be 9. Considering various–

possibilities for $C(b)$, it is not hard to verify that $N(A) = A\lambda\{b\}$. This is impossible, as was shown above. Suppose that the maximal intersection D of subgroups from $\mathfrak{M}_1$ is noncyclic. Then $|D| = 4$, and $N(D) \approx S_4$. This again contradicts the assumption. Thus, $|D| = 2$, if $\mathfrak{M}_2 \cup \mathfrak{M}_3 \neq \emptyset$. It turns out that the latter condition can be omitted. Moreover, $N(D) \in \mathfrak{M}_2 \cup \mathfrak{M}_3$.

Applying the first theorem of Grün, we find in G a subgroup G_0 of index 2. Since $K \cap G_0 = S\lambda A$, the Sylow 2-subgroups in G_0 are pairwise mutually disjoint. From (2) it follows that G_0 is isomorphic to $SL(2, q)$ or $Sz(q)$, where q is a power of 2. But since the rank of the center S is equal to 2, the order of S must be 4. Therefore T is a dihedral group, which contradicts the assumption. The lemma is proved.

2nd case. $\mathfrak{M}_1 = \emptyset$, $\mathfrak{M}_2 \neq \emptyset$. Denote by D the maximal intersection of any two distinct subgroups from $\mathfrak{M}_2$. Since in groups (of type 2) from $\mathfrak{M}_2$ all elements outside the Fitting radical are 2-elements, D is a 2-group isomorphic to a subgroup of a direct product of two 2-groups, each of which has a unique involution. In particular, the rank of the center of D is not greater than 2. Suppose that D is a Sylow 2-subgroup in G. Since the lower layer D_0 of the center of D is weakly closed in D with respect to G, the greatest 2-factor group of the group G is isomorphic to the same factor group for $N(D_0)$. From $C(D_0) = D$ we have $N(D_0) = D\lambda\{b\}$, a Frobenius group, where $b^3 = e$. It is easy to check that if a subgroup of a direct product of 2-groups, each of which has a unique involution, admits a regular automorphism of order 3, then this subgroup is a direct product of isomorphic cyclic groups. From the nonsolvability of G it must then follow, as Brauer showed in (3), that $|D| = 4$.

Suppose next that in G there are no self-centralizing subgroups of order 4. Thus, D cannot be a Sylow 2-subgroup in G, and a Sylow 2-subgroup T of G coincides with its normalizer. Now it is easy to verify that D is a maximal intersection of Sylow 2-subgroups of the group G, and either $N(D) \in \mathfrak{M}_3$ (in this case D has a unique involution), or $N(D) = N(D_0)$ is a CJT-group of type 4, $N(D)/D = S_3$. Hence the center of a Sylow 2-subgroup of G is cyclic, and G is not 2-normal. Therefore there exists such a D that $N(D) \notin \mathfrak{M}_3$. A Sylow 2-subgroup of $N(D)$ is a wreath product of two cyclic groups: of order 2^m, $m > 1$, and of order 2. It coincides with its normalizer, and therefore is a Sylow 2-subgroup in G. By the first theorem of Grün, G has a subgroup G_0 of index 2. The Sylow 2-subgroup in G_0 is D. From (3) it follows that G_0, and hence also G, is solvable. This contradicts the assumption. Taking Lemma 3 into account, we summarize the preceding arguments. We shall call a 2-group T semidihedral if

$$T = \{a, b\}, \quad a^{2^{n+1}} = b^2 = e, \quad b^{-1}ab = a^{-1+2^n}, \quad n \geqslant 2.$$

Lemma 4. *A Sylow 2-subgroup of a nonsolvable QP-, but not CJT-group is dihedral or semidihedral if $\mathfrak{M}_1 \cup \mathfrak{M}_2 \neq \emptyset$.*

3rd case: $\mathfrak{M}_1 = \mathfrak{M}_2 = \emptyset$, $\mathfrak{M}_3 \neq \emptyset$. Suppose also that in G there are no self-centralizing subgroups of order 4. Let $M \in \mathfrak{M}_3$, and let D be the maximal intersection of M with subgroups from $\mathfrak{M}_3$ distinct from M. D is a Sylow 2-subgroup in M, and the number of involutions in D is 3. The center of a Sylow 2-subgroup T of G is therefore cyclic, T coincides with its normalizer, and G is not 2-normal. Note also that the number of involutions in T is greater than 3. Denote by D_0 the lower layer of the center of D. $N(D_0) = N(D)$ and is a 2-subgroup, $|N(D) : D| = 2$, and the number of involutions in $N(D)$ is greater than three. Hence it follows that the involutions from M split into two classes of involutions conjugate in G: the class of involutions each of which is contained in the center of some Sylow 2-subgroup of G, and the class of the remaining involutions. $N(D)$, in turn,

is distinct from its normalizer, and the latter is also a 2-subgroup. Therefore a Φ-subgroup of $N(D)$ has only one involution. It is now easy to check that M has the following structure: $M = \{s\} \times (A\lambda P)$, where $s^2 = e$, $A\lambda P$ is a Frobenius group whose complement P is a 2-group, and whose invariant set A is an abelian

Hall subgroup of G. Since, by assumption, a Sylow 2-subgroup T of G is not a group of maximal class, T has an invariant noncyclic subgroup V_T of order 4 ([4]). Using the notion of a Thompson family of 2-subgroups and the method of ([1]), one can show that every involution in V_T is contained in the center of some Sylow 2-subgroup of G. Hence it follows that $C(V_T)$ is a maximal subgroup in T, the rank of the center of $C(V_T)$ is 2, $N(C(V_T)) = C(V_T)\lambda\{b\}\lambda\{s\}$ is a group of type 4, where $b^3 = s^2 = e$, $C(s) \in \mathfrak{M}_3$. In particular, the nilpotency class of $C(V_T)$ is at most 2. Now one can apply the first theorem of Grün. We note that $C(V_T)$ does not contain those involutions s for which $C(s) \in \mathfrak{M}_3$. Thus all the groups referred to in Grün' s theorem are contained in $C(V_T)$. Hence G has a subgroup of index 2. From this fact, Lemma 4, and the principal results of ([1,5–7]), the main result follows:

Theorem 3. *In order that an insoluble group G be a QP-group, it is necessary and sufficient that G either be a CNP-group, or be isomorphic to A_7 or $H(q)$.*

A CNP**-group** is a group in which the centralizer of every nonidentity element has a nilpotent splitting. Such groups are described in ([7]). A_7 is the alternating group of degree 7. By $H(q)$, following ([8]), is denoted a split extension of $PSL(2, q)$ by a group of order 2 whose Sylow 2-subgroup is semidihedral.

A QNP**-group** is a group in which the following condition is satisfied:

(QNP). The intersection of any two distinct non-invariant maximal nilpotent subgroups is equal to the identity subgroup.

Corollary. *An insoluble group is a QP-group if and only if it is a QNP-group.*

The description of the QNP-groups obtained by us will be published in another paper. The proofs, the outlines of which have been given here, largely repeat the arguments used in the description of QNP-groups.

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REFERENCES

1. M. Suzuki, Trans. Am. Math. Soc., **99**, No. 3, 425, 468 (1961).
2. M. Suzuki, Ann. Math., **80**, No. 1, 58 (1964).
3. R. Brauer, J. Algebra, **1**, No. 4, 317 (1964).
4. N. Blackburn, Proc. London Math. Soc., (3), **11**, No. 41, 3 (1961).
5. D. Gorenstein, J. H. Walter, Illinois J. Math., **6**, No. 4, 553 (1962).

6. . . , , **168**, No. 3 (1966).

7. . . , . , . . **29**, No. 3, 605 (1965).

8. W. J. Wong, J. Austr. Math. Soc., **4**, No. 1, 94 (1964).

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