ON AN ANALOGUE OF THE WEIERSTRASS CANONICAL PRODUCT FOR ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Abstract

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ON AN ANALOGUE OF THE WEIERSTRASS CANONICAL PRODUCT FOR ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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In the works of W. Stoll (^{1}) and P. Lelong (^{2}) it was shown that, in the presence of certain natural restrictions on the density of the distribution of the points of the analytic set

$$\mathfrak{M}_f = \{z = (z_1, \dots, z_n);\ f(z) = 0\}$$

of "zeros" of an entire function $f(z)$, there exists an entire function $\varphi(z)$ that vanishes at the points $z \in \mathfrak{M}_f$ and only at them and has, in a certain sense, minimal growth with respect to the aggregate of variables $z = (z_1, \dots, z_n)$. Such a function $\varphi(z)$, when certain normalization-type conditions are fulfilled, is naturally regarded as an analogue of the Weierstrass canonical product and called canonical. In (^{1,2}) it was also shown that $\ln \varphi(z)$, where $\varphi(z)$ is a canonical function, can be represented by means of an integral over the set $\mathfrak{M}_f$. In this connection, Stoll' s representation holds in a certain neighborhood of the origin that does not intersect $\mathfrak{M}_f$, while Lelong' s representation holds everywhere outside $\mathfrak{M}_f$.

In this note we consider the question of the existence and construction of canonical functions under the condition that not the whole set $\mathfrak{M}_f$ is specified, but only a certain, specially chosen $(n-1)$-dimensional part of it; moreover, alongside functions of minimal growth with respect to the aggregate of variables, functions having minimal growth in one of the variables are also studied.

For the precise formulation of the results we shall need the following definitions and notation:

1. By $d_n\varphi$ we denote the volume element in the space of real variables $\varphi_1, \cdots, \varphi_n$.

2. Let $f(z)$ be an entire function; $\tau = (\tau_1, \dots, \tau_n)$, where $\tau_1 > 0, \dots, \tau_n > 0$ and $\tau_1^2 + \dots + \tau_n^2 = 1$. Denote

$$\mathfrak{M}_{f,\tau} = \mathfrak{M}_f \cap \left\{ z;\ \frac{|z_1|}{\tau_1} = \dots = \frac{|z_n|}{\tau_n} \right\};$$

$$n_{f,\tau}(t) = \frac{1}{(2\pi)^{n-1}\tau_1\tau_2\cdots\tau_n} \int_0^{2\pi} \cdots \int_0^{2\pi} n_{f,\tau}(t;\varphi_1,\dots,\varphi_{n-1})\, d_{n-1}\varphi,$$

where $n_{f,\tau}(t;\varphi_1,\dots,\varphi_{n-1})$ is the number of zeros of the function

$$f(w\tau_1 e^{i\varphi_1}, \dots, w\tau_{n-1}e^{i\varphi_{n-1}}, w\tau_n)$$

in the disk $|w| \leq t$.

3. Let $f(z)$ be an entire function, $r^* = (r_1, \dots, r_{n-1})$, $r_1 > 0, \dots, r_{n-1} > 0$. Denote

$$\mathfrak{M}_{f,r^*} = \mathfrak{M}_f \cap \{z;\ |z_1| = r_1, \dots, |z_{n-1}| = r_{n-1},\ |z_n| < \infty\};$$

$$n_{f,r^*}(t) = \frac{1}{(2\pi)^{n-1}} \int_0^{2\pi} \cdots \int_0^{2\pi} n_{f,r^*}(t;\varphi_1,\dots,\varphi_{n-1})\, d_{n-1}\varphi,$$

where $n_{f,r^*}(t;\varphi_1,\dots,\varphi_{n-1})$ is the number of zeros of the function

$$f(r_1 e^{i\varphi_1}, \dots, r_{n-1}e^{i\varphi_{n-1}}, z_n)$$

in the disk $|z_n| \leq t$.

4. We shall call the number $\tilde{\rho}_f$, defined by the equality

$$\tilde{\rho}_f = \overline{\lim_{t\to\infty}} \frac{\ln n_{f,\tau}(t)}{\ln t}, \tag{1}$$

the **order of the function** $n_{f,\tau}(t)$. The order $\tilde{\rho}_{f,n}$ of the function $n_{f,r^*}(t)$ is defined analogously. The order ρ_f of the entire function $f(z)$ with respect to the aggregate of variables and the order $\rho_{f,n}$ of the function $f(z)$ with respect to the variable z_n are defined in the same way, replacing in (1) the function $n_{f,\tau}(t)$, respectively, by the functions $\ln M_f(t_1, \dots, t)$ and $\ln M_f(r_1, \dots, r_{n-1}, t)$, where

$$M_f(r_1, \dots, r_n) = \max_{|z_i|=r_i,\ i=1,\dots,n} |f(z)|.$$

We note that $\tilde{\rho}_f$ does not depend on the choice of τ, while $\tilde{\rho}_{f,n}$ and $\rho_{f,n}$ do not depend on the choice of r^*.

5. To shorten the notation we shall also write

$$z^* = (z_1, \dots, z_{n-1}), \qquad z = (z^*, z_n), \qquad \zeta^* = (\zeta_1, \dots, \zeta_{n-1}),$$

$$k = (k_1, \dots, k_n), \qquad |k| = k_1 + \dots + k_n, \qquad z^k = z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n},$$

$$\frac{z}{\zeta} = \left(\frac{z_1}{\zeta_1}, \dots, \frac{z_n}{\zeta_n}\right), \qquad C(r^*) = \{z^*;\ |z_1| \le r_1, \dots, |z_{n-1}| \le r_{n-1}\},$$

$$K_q(R; \nu(t)) = R^q \left\{ \int_0^R \frac{\nu(t)}{t^{q+1}}\, dt + R \int_R^\infty \frac{\nu(t)}{t^{q+2}}\, dt \right\}.$$

Theorem 1. Let the entire function $f(z)$ be such that $f(0) \neq 0$ and $\tilde{\rho}_f < \infty$. Then it can be represented in the form

$$f(z) = e^{g(z)} \varphi(z) e^{P_q(z)},$$

where the functions $g(z)$ and $\varphi(z)$ are entire, and

$$P_q(z) = \sum_{0 \le |k| \le q} \frac{z_1^{k_1} \cdots z_n^{k_n}}{k_1! \cdots k_n!} \left(\frac{\partial^{|k|} \ln f(z)}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} \right)_{z=0}.$$

Moreover:

1. The function $g(z)$ is defined by the equality

$$g(z) = -\frac{2}{(2\pi)^{n-1}} \lim_{R \to \infty} \int_0^{2\pi} \cdots \int_0^{2\pi} \ln |f(\zeta)|\, F_q\left(\frac{z}{\zeta}\right) d_n\varphi,$$

where

$$\zeta = (R\tau_1 e^{i\varphi_1}, \dots, R\tau_{ne}^{i\varphi_n}), \qquad F_q(z) = \sum_{|k| > q} z^k,$$

and q is the smallest integer for which the integral

$$\int^\infty \frac{n_{f,\tau}(t)}{t^{q+2}}\, dt$$

converges.

2. The function $\ln\varphi(z)$ in every polycylinder $\{|z_i| \le R\tau_i,\ i = 1,\dots,n\}$ not intersecting $\mathfrak{M}_f$, can be represented in the form

$$\ln\varphi(z) = -\frac{1}{(2\pi)^{n-1}}\int_{\mathfrak{M}_{f,\tau}} \Phi_q\left(\frac{z}{\zeta}\right) d_{n-1}\Phi,$$

where

$$\zeta = (w\tau_1 e^{i\varphi_1},\dots,w\tau_{n-1}e^{i\varphi_{n-1}}, w\tau_n), \qquad \Phi_q(z) = \sum_{|k|>q}\frac{z^k}{|k|}.$$

3. For every $\delta > 1$ there exists a constant $C_\delta > 0$ such that for every $R > 0$

$$\ln M_f(R\tau_1,\dots,R\tau_n) \le C_\delta K_q(R\delta;\, n_{f,\tau}(t))^{*}.$$

4. The functions $\varphi(z)$ and $g(z)$ do not depend on the choice of the vector τ.

Theorem 2. Let the entire function $f(z)$ be such that $\tilde\rho_f < \infty$, and let $\tilde\rho_{f,n}$ be either nonintegral, or $\tilde\rho_{f,n} = q$, where q is the smallest integer for which— for which the integral* converges,

$$\int^{\infty} \frac{n_{f,r^*}(t)}{t^{q+2}}\,dt.$$

Suppose, further, that $f(z^*, z_n) \ne 0$ for no fixed z^*. Then the function $f(z)$ can be represented in the form

$$f(z) = e^{g(z)}\psi(z),$$

where $g(z)$ is an entire function, and the function $\psi(z)$ is such that $\rho_{\psi,n} = \tilde\rho_{f,n}$. Moreover:

1. The function $g(z)$ in each polycylinder $\{z;\ z^* \in C(r^*);\ |z_n| < \infty\}$ is representable in the form

$$g(z) = \lim_{r_n\to\infty}\frac{2}{(2\pi)^{n-1}}\int_0^{2\pi}\cdots\int_0^{2\pi}\ln|f(\zeta)|\,F_q\left(\frac{z}{\zeta}\right) d_n\varphi,$$

where $\zeta = (r_1e^{i\varphi_1},\dots,r_{ne}^{i\varphi_n})$,

$$F_q(z) = z_n^{q+1}/(1-z_1)\cdots(1-z_n).$$

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* Hence, in particular, it follows that $\rho_\varphi = \tilde\rho_\varphi = \tilde\rho_f$.

2. The function $\psi(z)$ in any polycylinder

$$E = \{z;\ z^* \in C(r^*),\ |z_n| < \infty\}$$

such that $C(r^*) \subset \{z^*;\ f(z^*, 0) \neq 0\}$ is representable in the form

$$\psi(z) = \varphi(z) \exp\left\{ \sum_{m=0}^{q} \frac{z_n^m}{m!} \left(\frac{\partial^m \ln f(z)}{\partial z_n^m} \right)_{z_n=0} \right\},$$

where the function $\varphi(z)$, holomorphic in the polycylinder E, is determined in the polycylinder

$$E(r_n) = E \cap \{|z_n| \leq r_n\}$$

for sufficiently small r_n by the equality

$$\ln \varphi(z) = -\frac{1}{(2\pi)^{n-1}} \int_{\mathfrak{M}_{f,r^*}} \Phi_q\left(\frac{z^*}{\zeta^*}, \frac{z_n}{\zeta_n} \right) d_{n-1}\varphi$$

with

$$\zeta^* = (r_1 e^{i\varphi_1}, \dots, r_{n-1} e^{i\varphi_{n-1}})$$

and

$$\Phi_q(z) = \frac{1}{(1 - z_1) \cdots (1 - z_{n-1})} \sum_{m=q+1}^{\infty} \frac{z_n^m}{m}.$$

3. For every $\delta > 1$ there exists a constant $C_\delta > 0$ such that, for any $r_n > 0$ and r^* such that

$$C(r^*) \subset \{z^*;\ f(z^*, 0) \neq 0\},$$

the inequality holds

$$\ln M_f(r^*, r_n) \leq C_\delta K_q(r_n;\, n_{f,\delta r^*}(t)).$$

The proof of these theorems, which we omit here, is essentially based on the following theorems, which, in our view, are of independent interest:

Theorem 3. Let the entire function $f(z)$ be such that $\tilde{\rho}_f < \infty$. Suppose, further, that q is the least integer for which the integral

$$\int^{\infty} \frac{n_{f,\sigma}(t)}{t^{q+2}}\, dt$$

converges, and let $w_j(\lambda_1, \dots, \lambda_n)$, $j = 1, 2, \dots$, denote the zeros of the function

$$f(\lambda_1 w, \dots, \lambda_n w)$$

for fixed $\lambda_1, \lambda_2, \dots, \lambda_n$. Then, for any $m > q$, the sum

$$\sum_{j=1}^{\infty} w_j^{-m}(\lambda_1, \dots, \lambda_n)$$

is a homogeneous polynomial of degree m in the variables $\lambda_1, \dots, \lambda_n$.

Theorem 4. Let the entire function $f(z)$ be such that $\tilde{\rho}_f < \infty$. Suppose, further, that $z_{n,j}(z^*)$, $j = 1, 2, \dots$, are the zeros of the function $f(z^*, z_n)$ for fixed z^*. Then, for any $m > \tilde{\rho}_{f,n}$, the function

$$\frac{1}{m!} \left. \frac{\partial^m \ln f(z)}{\partial z_n^m} \right|_{z_n=0} + \frac{1}{m} \sum_{j=1}^{\infty} Z_{n,j}^{-m}(z^*),$$

defined in the domain $\{z^*;\ f(z^*, 0) \neq 0\}$, extends to all of C^{n-1} as an entire function.

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CITED LITERATURE

1. W. Stoll, Math. Zs., 57, 211 (1953).
2. P. Lelong, J. Anal. Math., 12, 365 (1965).
3. L. I. Ronkin, DAN, 172, No. 5 (1967).

* As follows from (3), q does not depend on the choice of the vector r^*.

Note: Figure translations are in progress. See original paper for figures.