ON A CLASS OF INVERSE BOUNDARY-VALUE PROBLEMS WITH UNKNOWN COEFFICIENTS

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ON A CLASS OF INVERSE BOUNDARY-VALUE PROBLEMS WITH UNKNOWN CO-EFFICIENTS

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Recently, increasing attention has been devoted to inverse problems in which the unknowns are the coefficients of differential equations. Various formulations are possible in this connection. In the present note one class of inverse boundary-value problems is considered for an equation of parabolic type and for an ordinary differential equation. In the parabolic case the unknown coefficients may depend on x and t, while in the case of an ordinary equation they depend on x; moreover, in both cases the dependence on x is piecewise constant. Uniqueness and stability of the solution are established when additional data are prescribed at a finite number of points, and the convergence of an iteration method and of a difference method is also investigated.

1°. **A boundary-value problem with unknown coefficients for an equation of parabolic type.** Suppose it is required to find the triple of functions $\{a(x,t), c(t), u(x,t)\}$ from the conditions

$$a(x,t)u_{xx} - c(t)b(x)u - u_t = H(x,t,a(x,t),c(t)), \quad l_{k-1} < x < l_k, \\ k = 1, \dots, n, \quad 0 < t \le T; \tag{1}$$

$$u(0,t) = f_0(t), \quad u(l,t) = f_n(t), \quad 0 \le t \le T; \tag{2_1}$$

$$u(l_k - 0, t) = u(l_k + 0, t), \quad a(l_k - 0, t)u_x(l_k - 0, t) = \\ = a(l_k + 0, t)u_x(l_k + 0, t), \quad k = 1, \dots, n-1, \\ 0 = l_0 < l_1 < \dots < l_{n-1} < l_n = l, \quad 0 \le t \le T; \tag{2_2}$$

$$u(x,0) = \varphi(x), \quad 0 \le x \le l, \quad \varphi(0) = f_0(0), \quad \varphi(l) = f_n(0); \tag{3}$$

$$-a(0,t)u_x(0,t) = g(t) > 0, \quad -a(l,t)u_x(l,t) = c(t)\psi(t), \quad \psi(t) > 0,$$
$$0 \le t \le T; \tag{4}$$

$$u(l_k,t) = f_k(t), \quad \varphi(l_k) = f_k(0), \quad k = 1, \dots, n-1, \tag{5}$$

where $H(x,t,a,c)$ is given and continuous in $\Pi\{0 \le x \le l,\ 0 \le t \le T,\ 0 \le a \le A,\ 0 \le c \le C\}$, while $f_k(t)$, $k = 0, 1, \dots, n$, $g(t)$, $\psi(t)$, and $\varphi(x)$ are given and continuous for $0 \le t \le T$ and $0 \le x \le l$, respectively; $b(x)$ is given, and $a(x,t)$ is the unknown function, piecewise constant in x, with discontinuities only at the points $x = l_k,\ k = 1, \dots, n-1$.

Definition 1. A triple of functions $\{a(x,t), c(t), u(x,t)\}$ will be called a **solution of problem** (1)–(5) if these functions satisfy the following requirements: 1) $a(x,t)$ is a continuous function of t for $0 \le t \le T$, piecewise constant in x for $0 \le x \le l$, having discontinuities only at $x = l_k,\ k = 1, 2, \dots, n-1$, and $a(x,t) > 0$; 2) $c(t)$ is a continuous function of t for $0 \le t \le T$, $c(t) \ge 0$; 3) $u(x,t)$ is continuous in $\overline{D}\{0 \le x \le l,\ 0 \le t \le T\}$; $u_x(x,t)$, $u_{xx}(x,t)$, $u_t(x,t)$ are defined and continuous in

$D_k\{l_{k-1} < x < l_k,\ 0 < t \le T\},\ k = 1, \dots, n$, and the limits $v_x(l_k - 0, t)$, $u_x(l_k + 0, t)$ exist; 4) all relations (1)–(5) are satisfied.

Let us consider the **uniqueness** and **stability** of the solution of problem (1)–(5). Denote by $a_k(t)$, b_k, and $u_k(x,t)$ the values of $a(x,t)$, $b(x)$, and $u(x,t)$ in the domains $\overline{D}_k\{l_{k-1} \le x \le l_k,\ 0 \le t \le T\}$, $k = 1, \dots, n$. Suppose that, along with problem (1)–(5), another problem $(\overline{1})$–$(\overline{5})$ is given, differing from problem (1)–(5) in that $a_k, c, u_k, H, \varphi, f_{k-1}, f_k, g, \psi$ are replaced by $\overline{a}_k, \overline{c}, \overline{u}_k, \overline{H}, \overline{\varphi}, \overline{f}_{k-1}, \overline{f}_k, \overline{g}, \overline{\psi}$, and k takes the values indicated above. Put $\delta_H = \overline{H} - H,\ \delta_\varphi = \overline{\varphi} - \varphi,\ \delta_{b_k} = \overline{b}_k - b_k$

$$\delta_{f_k} = \overline{f}_k - f_k, \quad k = 0, 1, \dots, n; \qquad \delta_g = \overline{g} - g, \quad \delta_\psi = \overline{\psi} - \psi; \tag{6}$$

$$z_k = \overline{u}_k(x,t) - u_k(x,t), \qquad \lambda_k(t) = \overline{a}_k(t) - a_k(t), \qquad \mu(t) = \overline{c}(t) - c(t),$$

$$k = 1, \dots, n. \tag{7}$$

We shall assume that the "perturbations" $\delta_H, \delta_{f_l}, \delta_\varphi, \delta_g, \delta_\psi$ are continuous together with the derivatives $(\delta_\varphi)'_x$, $(\delta_\varphi)''_{xx}$, $(\delta_{f_k})'_t$. Such perturbations $\delta_H, \delta_\varphi, \delta_{f_l}, \delta_g, \delta_\psi$ will be called **admissible**. We shall say that a solution of problem (1)–(5) belongs to the class $C_{0021}(\overline{D})$ if $a_k(t) \in C_0[0,T]$,

$c(t) \in C_0[0,T]$, $u_k(x,t) \in C_{21}(\overline{D}_k)$, $k = 1, \dots, n$, where $\overline{D}_k$ is the closed domain $\{l_{k-1} < x < l_k,\ 0 < t < T\}$.

Definition 2. If for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that, for any admissible $\delta_H, \delta_\varphi, \delta_{f_l}, \delta_{b_k}, \delta_g, \delta_\psi$ satisfying the conditions

$$|\delta_H| < \delta, \quad |\delta_\varphi| < \delta, \quad |\delta_{f_k}| < \delta, \quad |\delta_{b_k}| < \delta, \quad |\delta_g| < \delta, \quad |\delta_\psi| < \delta, \quad |(\delta_\varphi)'_x| < \delta,$$

$$|(\delta_\varphi)''_{xx}| < \delta, \quad |(\delta_{f_k})'_t| < \delta \tag{8}$$

the inequalities

$$|z_k(x,t)| < \varepsilon \text{ in } \overline{D}, \qquad |\lambda_k(t)| < \varepsilon, \qquad |\mu(t)| < \varepsilon \quad \text{for } 0 \le t \le T,$$

$$k = 1, \dots, n, \tag{9}$$

hold for $\{a(x,t), c(t), u(x,t)\}$ and $\{\overline{a}(x,t), \overline{c}(t), \overline{u}(x,t)\}$ from $C_{0021}(\overline{D})$, then we shall say that the solution of problem (1)–(5) is **stable** in the class $C_{0021}(\overline{D})$ with respect to admissible perturbations of its data.

Theorem 1. *If $H,\ H_x,\ H_a,\ H_c,\ \varphi,\ \varphi_x,\ \varphi_{xx},\ f_{k-1},\ f_{k-1,t},\ f_k,\ f_{kt},\ g,\ \psi$ are continuous and bounded functions of all their arguments; $u_{kx}(l_{k-1},t) \ne 0,\ k = 1,2,\dots,n,$ and $\psi(t) \ne 0,\ 0 \le t \le T$, then the solution of problem (1)–(5) is unique and stable in the class $C_{0021}(\overline{D})$ with respect to admissible perturbations of its data.*

Problem (1)–(5) can be solved by the method of iteration according to the formulas

$$a_k^{(s)}(t)u_{kxx}^{(s)} - c^{(s)}(t)b_k u_k^{(s)} = u_{kt}^s H(x,t,a_k^{(s)},c^{(s)}), \quad l_{k-1} < x < l_k, \quad 0 \le t \le T; \tag{10}$$

$$u_k^{(s)}(l_{k-1},t) = f_{k-1}(t), \qquad u_k^{(s)}(l_k,t) = f_k(t), \quad 0 \le t \le T, \quad k = 1, \dots, n; \tag{11}$$

$$u_k^{(s)}(x,0) = \varphi(x), \qquad l_{k-1} \le x \le l_k, \quad k = 1, \dots, n; \tag{12}$$

$$-a_1^{(s+1)}(t)u_{1x}^{(s)}(0,t) = g(t), \qquad 0 \le t \le T; \tag{13}$$

$$a_{k+1}^{(s+1)}(t)u_{k+1,x}^{(s)}(l_k,t)=a_k^{(s+1)}(t)u_{kx}^{(s)}(l_k,t),\quad k=1,\dots,n-1,\quad 0\le t\le T; \tag{14}$$

$$-a_n^{(s+1)}(t)u_{nx}^{(s)}(l,t)=c^{(s+1)}(t)\psi(t),\qquad \psi(t)>0,\quad 0\le t\le T, \tag{15}$$

under the condition that

$$u_{kx}^{(s)}(l_{k-1},t)\ne 0\quad \text{for } k=1,\dots,n,\quad 0\le t\le T. \tag{16}$$

Theorem 2. Suppose that a solution of problem (1)–(5) exists, belongs to the class $C_{1121}(\overline{D})$, and suppose that the following conditions are satisfied:

a) $f_k(t)\geqslant 0,\ k=1,\dots,n-1,\ f_n(t)\equiv 0,\ f_{k-1}(t)-f_k(t)>0,\ \varphi(x)\geqslant 0,\ \varphi_x(x)\leqslant 0,\ \varphi_x(0)<0,\ \varphi_{xx}(x)\geqslant 0,\ H(l_{k-1},t,a_k,c)-f_{k-1,t}(t)\geqslant 0,\ H\leqslant 0,\ H_{xx}\leqslant 0,\ H(l,t,a_n,c)\equiv 0$;

b) the zero-order compatibility conditions $f_k(0)=\varphi(l_k),\ k=0,1,\dots,n$, and the first-order compatibility conditions: $a_k(0)\varphi_{xx}(l_{k-1})-c(0)b_k\varphi(l_{k-1})-f_{k-1,t}(0)=H(l_{k-1},0,a_k(0),c(0)),\ a_k(0)\varphi_{xx}(l_k)-c(0)b_k\varphi(l_k)-f_{kt}(0)=H(l_k,0,a_k(0),c(0))$, where $a_k(0)=g(0)[-\varphi_x(0)]^{-1},\ c(0)=g(0)\varphi_x(l)[\psi(0)\varphi_x(0)]^{-1}$;

c) $H,\ \varphi$ and their derivatives up to fourth order with respect to x, as well as $H_t,\ H_a,\ H_c,\ f_k(t),\ f_{kt},\ f_{ktt},\ g(t),\ g_t(t),\ \psi(t),\ \psi_t(t)$, are continuous and bounded in their domains of definition.

Then the successive approximations obtained by formulas (10)–(15) converge uniformly with rate $M_1M_2^s[s!]^{-1/2}$ as $s\to+\infty$ to the solution of problem (1)–(5), where the constants M_1 and M_2 depend on the data of the problem.

2°. A boundary-value problem with unknown coefficients for an ordinary differential equation.

Suppose it is required to find $a(x), c$ and $u(x)$ from the conditions

$$a(x)u_{xx}(x)-cb(x)u(x)=H(x,a(x),c),\quad 0<x<l,\quad x\ne l_k,\\ k=1,\dots,n-1; \tag{17}$$

$$u(l_k)=f_k,\quad k=0,1,\dots,n,\quad 0=l_0<l_1<\dots<l_{n-1}<l_n=l; \tag{18}$$

$$u(l_k-0)=u(l_k+0),\quad a(l_k-0)u_x(l_k-0)=a(l_k+0)u_x(l_k+0),\\ k=1,\dots,n-1, \tag{19}$$

$$-a(0)u_x(0) = g, \quad g > 0, \quad -a(l)u_x(l) = c\psi, \quad \psi > 0, \tag{20}$$

where $H(x, a, c)$ is a given continuous function in $\overline{\Pi}\{0 \leqslant x \leqslant l,\ 0 \leqslant a \leqslant A,\ 0 \leqslant c \leqslant C\}$; $f_k,\ g,\ \psi$ are given constants; $b(x)$ is given and $a(x)$ is the sought piecewise-constant function of $x,\ 0 \leqslant x \leqslant l$, having discontinuities only at the points $x = l_k,\ k = 1, \dots, n-1$; c is the sought constant.

Definition 3. A triple of quantities $\{a(x), c, u(x)\}$ will be called a **solution of problem** (17)–(20) if: 1) $a(x) > 0$ for $0 \leqslant x \leqslant l$; $a(x)$ is a piecewise-constant function having discontinuities only at the points $x = l_k,\ k = 1, \dots, n-1$; 2) $c > 0$, c is a constant (the case $c = 0$ is of no interest); 3) $u(x)$ is continuous in $\overline{D}\{0 \leqslant x \leqslant l\}$, has continuous derivatives $u_x(x)$ and $u_{xx}(x)$ in $D_k\{l_{k-1} < x < l_k\},\ k = 1, 2, \dots, n$, and the limits $u_x(l_k + 0),\ u_x(l_k - 0)$ exist; 4) all relations (17)–(20) are satisfied.

Denote by $a_k,\ b_k$ and $u_k(x)$ the values of $a(x),\ b(x)$ and $u(x)$ in the domains $\overline{D}_k\{l_{k-1} \leqslant x \leqslant l_k\}$.

Suppose that, along with problem (17)–(20), there is given a problem $(\overline{17})$—$(\overline{20})$, differing from problem (17)–(20) only in that the quantities $a_k,\ c,\ u_k,\ H,\ f_k,\ g,\ \psi$ are replaced by the quantities $\overline{a}_k,\ \overline{c},\ \overline{u}_k,\ \overline{H},\ \overline{f}_k,\ \overline{g},\ \overline{\psi}$. Put

$$z_k(x) = \overline{u}_k(x) - u_k(x), \quad \lambda_k = \overline{a}_k - a_k, \quad \mu = \overline{c} - c.$$

We shall assume that δ_H is a continuous function of all its arguments; $\overline{u}_k(x) \in C_2(\overline{D}_k)$ and $\overline{u}_k(x) \in C_2(\overline{D}_k),\ k = 1, \dots, n$.

Definition 4. If for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that, for all $\delta_H,\ \delta_{f_i},\ \delta_g,\ \delta_\psi$ satisfying the inequalities

$$|\delta_H| < \delta, \quad |\delta_{f_i}| < \delta, \quad |\delta_g| < \delta, \quad |\delta_\psi| < \delta,$$

the inequalities

$$|z_k(x)| < \varepsilon \quad \text{for } l_{k-1} \leqslant x \leqslant l_k,\ k = 1, \dots, n, \quad |\lambda_k| < \varepsilon, \quad |\mu| < \varepsilon$$

hold, then the solution $a_k(x),\ c,\ u_k(x)$ of problem (17)–(20) is called **stable with respect to perturbations of its data**.

Theorem 3. If

$$\left|u_{k+1,x}(l_k)\right| \geq d > 0, \qquad k = 0, 1, \dots, n-1,$$

$$\max\left\{\int_{l_{k-1}}^{l_k} |H_a - u_{kxx}|\ dx,\ \int_{l_{k-1}}^{l_k} |H_c + b_k u_k|\ dx\right\} \leq$$

$$\leq \min\{d, |\psi|\}\frac{q}{n+1}, \qquad |u_{kx}(l_k)| \leq |u_{k+1,x}(l_k)|, \qquad |u_{xx}(l)| \leq \psi,$$

H, H_x, H_a, H_c are continuous in $\overline{\Pi}$, then the solution of problem (17)–(20) is unique and stable with respect to perturbations of its data.

We shall seek the solution of problem (17)–(20) with the aid of the difference-iteration scheme

$${}^s a_k^h \delta_{xx} {}^s u_i^h - {}^s c^h b_k {}^s u_i^h = {}^s H_i^h = H\left(x_i, {}^s a_i^h, {}^s c^h\right),$$

$$M_{k-1} + 1 \leq i \leq M_k - 1, \quad k = 1, \dots, n; \tag{17_{ki}^s}$$

$${}^s u_{M_{k-1}}^h = f_{k-1}, \qquad {}^s u_{M_k}^h = f_k; \tag{18_{ki}^s}$$

$${}^{s+1} a_{k+1}^h \delta_x {}^s u_{M_k}^h = {}^s a_k^h \delta_{\overline{x}} {}^s u_{M_k}^h; \tag{19_{ki}^s}$$

$$-{}^{s+1} a_1^h \delta_x {}^s u_0^h = g, \qquad -{}^{s+1} a_n^h \delta_{\overline{x}} {}^s u_{M_n}^h = {}^{s+1} c^h \psi. \tag{20_{ki}^s}$$

Theorem 4. Suppose that the following conditions are satisfied:

a) $0 \leq f_k \leq f_{\max}$, $k = 0, \dots, n$,

$$f_{k-1} > f_k \text{ ch}\left[\sqrt{\chi_k b_k \psi^{-1}}(l_k - l_{k-1})\right],$$

$k = 1, 2, \dots, n$, $\chi_k = (f_{k-1} - f_k)(l_k - l_{k-1})^{-1}$, $g > 0$, $\psi > 0$;

b) $H < 0$, $H_{xx} \leq 0$, $H_{xxxx} \leq 0$,

$$H(l_k, a_k, c) + c_0 b_k f_k > 0,$$

$$H_{xx}(l_k, a_k, c) + \chi_k \left[H(l_k, a_k, c) + c_0 b_k f_k\right] \geq 0,$$

$k = 0, 1, \dots, n$, $a_1 = qp^{-1}$, $a_{k+1} = a_k q_k p_{k+1}^{-1}$, where

$$\chi_n = q_n, \qquad \chi_k = \chi_{k+1} q_k p_{k+1}^{-1}, \qquad k = 1, 2, \dots, n-1,$$

$$p_k = \{(f_{k-1} - f_k)[1 + (2\psi)^{-1}(l_k - l_{k-1})^2 \chi_k] + (8A_k)^{-1} \|\delta_{xx}^4 H\|_0 (l_k - l_{k-1})^4\}(l_k - l_{k-1})^{-1},$$

$$q_k = (1-\varepsilon)\sqrt{\chi_k \psi^{-1} b_k}\left\{\operatorname{sh}\sqrt{\chi_k\psi^{-1}b_k}(l_k - l_{k-1})\right\}^{-1}$$

$$\times\left\{f_{k-1} - f_k \operatorname{ch}\sqrt{\chi_k\psi^{-1}b_k}(l_k - l_{k-1})\right\}, \qquad A_k = g(f_{k-1} - f_k)^{-1}(l_k - l_{k-1});$$

c) $H, H_x, H_{xx}, H_{xxx}, H_{xxxx}$ exist and are continuous in Π, moreover H, H_x, H_{xx} exist and are bounded in $\overline{\Pi}$. Suppose, further, that

$$0 < a_k \leq {}^0a_k^h \leq A_k,$$

$$0 < c_0 \leq {}^0c^h \leq C, \qquad \chi_k\psi^{-1} \leq {}^sc^h\left[{}^sa_k^h\right]^{-1} \leq \chi_k\psi^{1-}.$$

Then, for all sufficiently small $h > 0$ and $s \to +\infty$, the values ${}^sa_k^h$, ${}^sc^h$, ${}^su_i^h$ tend, with the rate of a geometric progression, to the values a_k^h, c^h, u_i^h, forming the solution of the corresponding difference problem, defined by relations (17_{ki}^s) $-(20_{ki}^s)$, if everywhere in these relations the iteration indices s and $s+1$ are removed; the solution a_k^h, c^h, u_i^h of the indicated difference problem, as $h \to 0$, converges to the solution a_k, c, u_k of problem (17)–(20), where $u_k(x) \in C_2(\overline{D}_k)$, $k = 1, \dots, n$.

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Note: Figure translations are in progress. See original paper for figures.