ON A METHOD OF FRONT STRAIGHTENING FOR MULTI-FRONT ONE-DIMENSIONAL PROBLEMS OF STEFAN TYPE

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ON A METHOD OF FRONT STRAIGHTENING FOR MULTI-FRONT ONE-DIMENSIONAL PROBLEMS OF STEFAN TYPE

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1°. Let it be required to find the functions $u = u(x,t)$, $\xi_i = \xi_i(t)$, $i = 1, 2, \ldots, N$, from the conditions

$$Lu \equiv u_t - a_i(x,t,u)u_{xx} + b_i(x,t,u,u_x,\xi_s,\xi_s') = 0,$$
$$(x,t) \in D_{iT}\{\xi_i(t) < x < \xi_{i+1}(t), \quad 0 < t < T\}, \quad i = 1, 2, \ldots, N-1; \quad (1)$$

$$u\big|_{x=\xi_i(t)} = U_i(\xi_i(t), t), \quad t > 0, \quad i = 1, 2, \ldots, N; \quad (2)$$

$$u\big|_{t=0} = U_i^0(x) \quad \text{for } x \in [\xi_i(0), \xi_{i+1}(0)], \quad i = 1, 2, \ldots, N-1; \quad (3)$$

$$l(\xi_i) \equiv \xi_i' - c_i(x,t,u)u_x\big|_{x=\xi_i(t)+0} + c_{i-1}(x,t,u)u_x\big|_{x=\xi_i(t)-0} - \quad (4)$$
$$-\Phi_i(x,t,u,u_x)\big|_{x=\xi_i(t)} = 0,$$

$$t > 0, \quad i = 1, 2, \ldots, N, \quad c_0(x,t,u) \equiv 0, \quad c_N(x,t,u) \equiv 0;$$

$$\xi_i\big|_{t=0} = \xi_{i0}, \quad \xi_{i0} = \text{const}, \quad i = 1, 2, \ldots, N, \quad (5)$$

where

$$\xi_i' = d\xi_i(t)/dt, \quad u_x\big|_{x=\xi_i(t)\pm 0} = \lim_{x \to \xi_i(t)\pm 0} u_x(x,t),$$

$$\Phi_i\big|_{x=\xi_i(t)} = \Phi_i(\xi_i(t), t, \ U_i, \ u_x\big|_{x=\xi_i-0}, \ u_x\big|_{x=\xi_i+0}).$$

We shall call (1)–(5) a **multi-front or multiphase Stefan-type problem with internal and external fronts**. If $\xi_1 = \xi_1(t)$ and $\xi_N = \xi_N(t)$ are known functions of t, then for $i = 1$ and $i = N$, instead of conditions (2) and (4), boundary conditions must be prescribed. Multi-front Stefan-type problems were considered in ([1–5]), and in ([1]) the existence and uniqueness of a generalized solution were first established.

In the present note the existence of a smooth solution of problem (1)–(5) on the interval $0 \le t \le T$ is established and a difference scheme is given under the assumptions that: 1) at $t = 0$ all N fronts ξ_i have already formed; 2) for $0 \le t \le T$ the number of fronts N does not change and they do not intersect. Restrictions on the data of problem (1)–(5) ensuring fulfillment of condition 2) will be indicated.

To prove the existence of a solution of (1)–(5) under the fulfillment of 1), 2), we make a change of independent variables (front straightening)

$$y_i = [\xi_{i+1}(t) - \xi_i(t)]^{-1}[x - \xi_i(t)], \quad t = t, \quad i = 1, 2, \dots, N-1. \tag{6}$$

Under this change the domains $\overline{D}_{iT} = \{\xi_i(t) \le x \le \xi_{i+1}(t),\ 0 \le t \le T\}$ pass into the domains $\overline{\Pi}_{iT} = \{0 \le y_i \le 1,\ 0 \le t \le T\}$, $i = 1, 2, \dots, N-1$, and problem (1)–(5) in the new variables takes the form

$$Lu \equiv u_t - (\xi_{i+1} - \xi_i)^{-2} a_i u_{y_i y_i} - (\xi_{i+1} - \xi_i)^{-1}[\xi_i' + y_i(\xi_{i+1}' - \xi_i')]u_{y_i} + b_i = 0 \tag{7}$$

for

$$(y_i, t) \in \Pi_{iT} = \{0 < y_i < 1,\ 0 < t < T\}, \quad i = 1, 2, \dots, N-1;$$

$$\begin{gathered} u\big|_{y_1=0+0} = U_1(\xi_1(t), t); \qquad u\big|_{y_{i-1}=1-0} = u\big|_{y_i=0+0} = U_i(\xi_i(t), t), \\ i = 2, 3, \dots, N-1; \qquad u\big|_{y_{N-1}=1-0} = U_N(\xi_N(t), t), \end{gathered} \tag{8}$$

$$u\big|_{t=0} = U_i^0 \quad \text{for } y_i \in [0, 1], \qquad i = 1, 2, \dots, N-1; \tag{9}$$

$$\begin{gathered} l(\xi_i) \equiv \xi_i' - c_i p_i\big|_{y_i=0+0} + c_{i-1} p_{i-1}\big|_{y_{i-1}=1-0} - \Phi_i\big|_{y_i=0} = 0, \\ i = 1, 2, \dots, N, \qquad t \geqslant 0; \qquad c_0 \equiv 0, \quad c_N \equiv 0; \end{gathered} \tag{10}$$

$$\xi_i\big|_{t=0} = \xi_{i0}, \qquad i = 1, 2, \dots, N. \tag{11}$$

Here $a_i = a_i[\xi_i + y_i(\xi_{i+1} - \xi_i), t, u]$, etc.

(7)–(11) is a nonlinear boundary-value problem whose solution, if it exists, will also be a solution of the original problem (1)–(5). Solvability of (7)–(11) in the class of smooth functions can be established by the method of successive approximations, using known results for equations of parabolic type (see [6]). The successive approximations $u^{(s)}$ in Π_{iT} $(i = 1, 2, \dots, N-1)$, $\xi_i^{(s)}$ $(i = 1, 2, \dots, N)$, $s = 0, 1, \dots$, of problem (7)–(11) are defined as follows. We specify $\xi_i^{(0)} = \xi_i^{(0)}(t)$ (for example, from the conditions $\xi_i^{(0)}(t) \equiv \xi_{i0}$ for $0 \leqslant t \leqslant T$); from (7)–(9), for $\xi_i = \xi_i^{(0)}$, $\xi_i' = (\xi_i^{(0)})'$, we find $u^{(0)} = u^{(0)}(x,t)$ in Π_{iT}; next, from

$$l(\xi_i^{(s)}) \equiv (\xi_i^{(s)})' - c_i^{(s-1)} p_i^{(s-1)}\big|_{y_i=0+0} + c_{i-1}^{(s-1)} p_{i-1}^{(s-1)}\big|_{y_{i-1}=1-0} - \Phi_i^{(s-1)}\big|_{y_i=0} = 0,$$
$$i = 1, 2, \dots, N,$$

for $s = 1$ we obtain $\xi_i^{(1)}$, $(\xi_i^{(1)})'$. In general, knowing $\xi_i^{(s)}$, $(\xi_i^{(s)})'$, from (7)–(9) with $\xi_i = \xi_i^{(s)}$, $\xi_i' = (\xi_i^{(s)})'$ we determine $u^{(s)}(x,t)$, and from the expression for $l(\xi_i^{(s)})$ one can then obtain $\xi_i^{(s+1)}$, $(\xi_i^{(s+1)})'$, etc. The unique solvability of (7)–(9) for known $\xi_i^{(s)}$, $(\xi_i^{(s)})'$ follows from [6]. Suppose that, for any number s and $0 \leqslant t \leqslant T$, the $\xi_i^{(s)}(t)$ satisfy the inequalities

$$(\alpha_i \text{ are prescribed numbers}, \quad \alpha_N \geqslant \sum_{k=0}^{N-1} \alpha_k)$$

$$\xi_1^{(s)}(t) \geqslant \alpha_0 > -\alpha_1/2, \qquad \xi_{i+1}^{(s)}(t) - \xi_i^{(s)}(t) \geqslant \alpha_i \geqslant 0 \quad (i = 1, 2, \dots, N-1),$$
$$\xi_N^{(s)}(t) \leqslant \alpha_N < \infty, \qquad |(\xi_i^{(s)})'| \leqslant r_i, \quad r_i = \text{const}, \tag{12}$$

and, in addition, the following **conditions A** are satisfied (cf. [6], Theorems 13, 14):

a) for $(x,t) \in D_{iT}$,

$$\sum_{k=0}^{i-1} \alpha_k \leqslant \xi_i \leqslant \alpha_N - \sum_{k=i}^{N-1} \alpha_k, \qquad |\xi_i'| \leqslant r_i, \quad r_i = \text{const},$$

and arbitrary u, $a_i(x,t,u) \geqslant 0$,

$$b_i(x,t,u,0,\xi_s,\xi_s')\, u \geqslant -b_{Ii} u^2 - b_{IIi}, \qquad b_{Ii}, b_{IIi} = \text{const} \geqslant 0;$$

b) in

$$G_{i1} = \left\{ (x,t) \in \overline{D}_{iT},\ \sum_{k=0}^{i-1} \alpha_k \leqslant \xi_i \leqslant \alpha_N - \sum_{k=i}^{N-1} \alpha_k,\ |\xi_i'| \leqslant r_i,\ |u| \leqslant M_{1i} = \max_{\overline{D}_{iT}} |u|,\ p_i = |u_x| \text{ in } \overline{D}_{iT} \text{ arbitr}\right.$$

the functions $a_i(x,t,u)$ are continuous and continuously differentiable with respect to x, u, and the functions $b_i(x,t,u,u_x,\xi_s,\xi_s')$ are continuous, continuously differentiable with respect to u, u_x, ξ_s', and satisfy the Hölder condition in x, ξ_s with exponents $\beta, \beta/2$, and the inequalities

$$a_i(x,t,u) \geqslant a_{i0} = \text{const} > 0, \quad |b_i|+(|a_i|+|\partial a_i/\partial x|+\partial a_i/\partial u)\,(1+p_i)^2 \leqslant \mu_i(1+p_i)^2,$$

$$\mu_i = \text{const} > 0, \qquad i = 1,2,\ldots,N-1;$$

c) in

$$G_{i2} = \left\{ (x,t) \in \overline{D}_{iT},\ \sum_{k=0}^{i-1} \alpha_k \leqslant \xi_i \leqslant \alpha_N - \sum_{k=i}^{N-1} \alpha_k,\ |\xi_i'| \leqslant r_i,\ |u| \leqslant M_{1i} = \max_{\overline{D}_{iT}} |u|,\ p \leqslant M_{2i} = \max_{\overline{D}_{iT}} |u_x| \right\}$$

the functions a_i, b_i satisfy the condition Hölder in t with exponent $\beta/2$, the functions c_i, Φ_i are continuous and satisfy a Hölder condition in ξ, t, u with exponent $\beta/2$, the functions Φ_i satisfy a Lipschitz condition in p_i, p_{i-1};

d) $U_i^0 = U_i^0(x) \in C_{2,\beta}$ in x, $\xi_{i0} \leq x \leq \xi_{i+1,0}$ $(i = 1,2,\ldots,N-1)$, the functions $U_i(\xi_i(t),t)$, for $0 \leq t \leq T$, $\sum_{k=0}^{i-1} \alpha_k \leq \xi_i \leq \alpha_N - \sum_{k=i}^{N-1} \alpha_k$, are continuous and continuously differentiable with respect to ξ_i, t, and $(U_i)_{\xi_i}, (U_i)_t$ satisfy a Hölder condition in ξ_i, t with exponent $\beta/2$, and the compatibility conditions of the initial and boundary data of zero and first order are fulfilled.

Theorem 1. *If conditions A and the conditions (12) of nonintersection of the curves $\xi_i^{(s)}(t)$ (for any number s) are fulfilled, then there exists at least one solution u, ξ_i of problem (1)–(5), such that $u(x,t) \in C_{2,1}^{\beta,\beta/2}(\bar{D}_{iT})$, $\xi_i(t) \in C_{1,\beta/2}$ in t, $0 \leq t \leq T$, and $\xi_i(t)$ satisfy conditions (12).*

We give one variant of restrictions on the coefficients of problem (1)–(5) which ensure fulfillment of the nonintersection conditions (12) required in theorem 1 (and also of condition 2)–see above). Let $U_i \equiv U_i(t)$, in G_{i1} $|b_i| \leq B_i$, $B_i = \text{const} > 0$, $|\Phi_i|_{y_i=0}| \leq f_i$, $f_i = \text{const} > 0$; moreover the inequalities $0 < \omega < \min_i a_{i0}$ must be satisfied,

$$\delta_i = \min\left\{ \left(\alpha_N - \sum_{k=0}^{i-1} \alpha_k\right)^{-1} (a_{i0} - \omega),\ \left(\alpha_N - \sum_{k=i}^{N-1} \alpha_k\right)^{-1} (a_{i-1,0} - \omega) \right\} \geq$$

$$\geq f_i + \max|c_i|\, g_{i,\text{l}} + \max|c_{i-1}|\, g_{i-1,\text{r}}, \tag{13}$$

$$i = 1, 2, \dots, N, \qquad c_0 = c_N \equiv 0,$$

$$g_{i,\mathrm{l}} = \left(\alpha_N - \sum_{k=0}^{i-1} \alpha_k\right) \max \Big\{ (B_i + \max_t |(U_i)_t|)\omega^{-1},$$

$$2D_i\alpha_i^{-1}\left(2\sum_{k=i+1}^{N-1} \alpha_k + \alpha_i\right)^{-1},\ (\alpha_N - \xi_{i+1,0})^{-1} \max |U_i^0(x)| \Bigg\},$$

$$g_{i,\mathrm{r}} = \left(\alpha_N - \sum_{k=i+1}^{N-1} \alpha_k\right) \max \Big\{ (B_i + \max_t |(U_{i+1})_t|)\omega^{-1},$$

$$2D_i\alpha_i^{-1}\left(2\sum_{k=0}^{i-1} \alpha + \alpha_i\right)^{-1},\ \xi_{i0}^{-1} \max |U_i^0(x)| \Bigg\},$$

$$D_i = \max_t |U_i(t)| + \max_t |U_{i+1}(t)|.$$

2°. The approximate solution of problem (1)–(5), reduced by the substitution (6) to the form (7)–(11), can practically be sought by the method of finite differences. In each of the domains $\bar{\Pi}_{iT} = \{0 \leq y_i \leq 1,\ 0 \leq t \leq T\}$ $(i = 1, 2, \dots, N-1)$ we introduce a grid of nodes $R_{i,h\tau} = \{(y_{i,k}, t_n);\ y_{i,k} = kh_i;\ k = 0, 1, \dots, K_i; h_i = 1/K_i;\ t_n = n\tau;\ n = 0, 1, \dots, S;\ S\tau = T\}$ with constant steps h_i in $\bar{\Pi}_{iT}$. On the grid $R_{h\tau} = \bigcup_{i=1}^{N-1} R_{i,h\tau}$, (7)–(11) is replaced by a system of difference equations approximating it. We give one of the possible difference schemes:

$$L_{h\tau}[w_{i,kn}] \equiv \delta_t^- w_{i,kn} - (\eta_{i+1,n} - \eta_{i,n})^{-2} a_{i,kn} \delta_{y_i\bar{y}_i} w_{i,kn} +$$

$$+(\eta_{i+1,n} - \eta_{i,n})^{-1}[\delta_t^- \eta_{i,n} + kh_i(\delta_t^- \eta_{i+1,n} - \delta_t^- \eta_{i,n})]\, \delta_{\bar{y}_i} w_{i,kn} + b_{i,kn} = 0,$$

- $u,\ u_x,\ u_{xx},\ u_t$ are defined and continuous in $\bar{D}_{iT}$ and satisfy in $\bar{D}_{iT}$ Hölder conditions in x and t with exponents $\beta,\ \beta/2$, respectively; the functions $\xi_i(t)$ are continuous for $0 \leq t \leq T$ and have continuous derivatives $\xi_i'(t)$ satisfying a Hölder condition in t with exponent $\beta/2$.

$$i = 1, 2, \ldots, N-1; \quad k = 1, 2, \ldots, K_i - 1; \quad n = 1, 2 \ldots; \tag{14}$$

$$w_{1,0n} = U_{1,n}; \quad w_{i-1,K_{i-1}n} = w_{i,0n} = U_{i,n}, \quad i = 2, 3, \ldots, N-1;$$

$$w_{N-1,K_{N-1}n} = U_{N,n}; \tag{15}$$

$$w_{i,k0} = U_i^0, \quad k = 0, 1, \ldots, K_i; \quad i = 1, 2, \ldots, N-1; \tag{16}$$

$$l_{h\tau}(\eta_{i,n}) \equiv \delta_t \eta_{i,n} - (\eta_{i+1,n-1} - \eta_{i,n-1})^{-1} c_{i,0n-1} \delta_{y_i} w_{i,0n-1} +$$

$$+(\eta_{i,n-1} - \eta_{i-1,n-1})^{-1} c_{i-1,K_{i-1}n-1} \delta_{\bar{y}_i} w_{i-1,K_{i-1}n-1} - \Phi_{i,0n-1} = 0, \tag{17}$$

$$i = 1, 2, \ldots, N; \quad n = 1, 2 \ldots; \tag{18}$$

$$\eta_{i0} = \xi_{i0}, \quad i = 1, 2, \ldots, N. \tag{19}$$

Here $w_{i,kn}, \eta_{i,n}$ are functions defined on $\overline{R}_{i,h\tau}$–approximate values of the functions u, ξ_i at the nodes $(y_{i,k}, t_n)$ of the domain $\overline{\Pi}_{i\tau}$. In (14)–(19) the following notation is used:

$$\delta_t w_{i,kn} = \tau^{-1}(w_{i,kn} - w_{i,kn-1}), \quad \delta_{y_i} w_{i,kn} = h_i^{-1}(w_{i,k+1n} - w_{i,kn}),$$

$$\delta_{\bar{y}_i} w_{i,kn} = h_i^{-2}(w_{i,k+1n} - 2w_{i,kn} + w_{i,k-1n}),$$

$$a_{i,kn} = a_i[\eta_{i,n} + kh_i(\eta_{i+1,n} - \eta_{i,n}),\, t_n,\, w_{i,kn}],$$

$$c_{i,0n-1} = c_i(\eta_{i,n-1},\, t_n,\, U_{i,n-1}),$$

$$b_{i,kn} = b_i[\eta_{i,n} + kh_i(\eta_{i+1,n} - \eta_{i,n}),\, t_n,\, w_{i,kn},\, \delta_{y_i} w_{i,kn}(\eta_{i+1,n} - \eta_{i,n})^{-1},$$

$$\eta_{s,n},\, \delta_t \eta_{s,n}], \quad c_{i-1,K_{i-1}n-1} = c_{i-1}(\eta_{i,n-1},\, t_n,\, U_{i,n-1}),$$

$$\Phi_{i,0n-1} = \Phi_i[\eta_{i,n-1},\, t_n,\, U_{i,n-1},\, (\eta_{i+1,n-1} - \eta_{i,n-1})^{-1}\delta_{y_i} w_{i,0n-1},$$

$$(\eta_{i,n-1} - \eta_{i-1,n-1})^{-1}\delta_{\bar{y}_{i-1}} w_{i-1,K_{i-1}n-1}], \quad U_{i,n-1} = U_i(\eta_{i,n-1}, t_n),$$

$$U_{i,n} = U_i(\eta_{i,n}, t_n).$$

For a possible method of solving problem (14)–(19), see [5].

Theorem 2. If conditions A and (13) are satisfied, with $b_i \equiv b_i(x,t,u,\xi_s)$, $\Phi_i \equiv \Phi_i(x,t,u)$, and also with the derivatives $(c_i)_{\xi_i}, (c_i)_u, (\Phi_i)_{\xi_i}, (\Phi_i)_u$, $\delta_{y_i\bar{y}_i} u_{ikn}$, $\delta_{\bar{t}y_i} u_{ikn}$ uniformly bounded, then the approximate solution $w_{i,kn}, \eta_{i,n}$, obtained by means of the scheme (14)–(19), converges to the solution of the Stefan problem (7)–(11) (or (1)–(5)), provided $h_i, \tau \to 0$; for sufficiently small $h_i \le h_{i0}$, the nonintersection conditions for $\eta_{i,n}$ will hold: $\eta_{1,n} \ge a_0 > -a_1/2$, $\eta_{i+1,n} - \eta_{i,n} \ge a_i > 0$ $(i = 1, 2, \dots, N-1)$, $\eta_{Nm} \le a_N < +\infty$; and the asymptotic order of the error of the method $z_{i,kn} = w_{i,kn} - u_{i,kn}$, $\zeta_{i,n} = \eta_{i,n} - \xi_{i,n}$, in determining the temperature u, as well as the position of the fronts ξ_i, will be $O(h + \tau)$.

3°. The method of item 1 is also applicable in the case when some of the curves ξ_i are known functions of t with the usual conjugation conditions fulfilled on them, and also when there are Verigin-type fronts.

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Note: Figure translations are in progress. See original paper for figures.