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Abstract

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MATHEMATICS

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ON THE METRIZABILITY OF A FINALLY COMPACT p-SPACE WITH A POINT-COUNTABLE BASE

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§ 1. A base $\sigma = \{U_\alpha\}$ of a space X is called point-countable if each point $x \in X$ belongs to no more than a countable number of elements of this base. A. S. Miščenko proved (^{3}) that a point-countable base σ of a bicompact X is necessarily countable, and the bicompact X itself, consequently, is metrizable. On the other hand, A. S. Miščenko (^{3}), and then Michael (^{4}), constructed an example of a finally compact, hereditarily strongly paracompact space X with a point-countable but uncountable base (consequently, these spaces are non-metrizable). A. V. Arhangel' skii introduced the class of p-spaces (see (^{1},^{2})) and proved that the class of paracompact p-spaces coincides with the class of spaces admitting perfect mappings onto metric spaces (^{1},^{2}).

The main result of the present note is

Main theorem 1*. *Let X be a finally compact p-space, and let $\sigma = \{U_\alpha\}$ be its point-countable base. Then this base σ is necessarily countable, and the space X itself is metrizable with a countable base.*

The proof of this proposition, in its methods, is close to my paper (^{6}), and especially important for us will be the following proposition, proved in (^{6}) (Lemma 2, p. 106):

Lemma 1. *Let X be a finally compact p-space; U an open subset of X of type F_σ. Then there exists a metrizable space Y with a countable base and a perfect mapping $f : X \to Y$ such that the set U is marked** under f, i.e. $f^{-1}fU = U$.*

In addition, we shall need the following quite simple proposition (see (^{5})).

Lemma 2. *Let σ be a point-countable base of the space X; F a separable*** subspace. Then the system*

$$\sigma_F = \mathcal{E}(U \in \sigma,\ U \cap F \neq \Lambda)$$

is countable.

Definitions (see ([7])). Suppose mappings $f_\lambda : X \to Y_\lambda$ are given from one and the same space X onto spaces Y_λ. Define the mapping

$$f : X \to Y \subseteq \prod_\lambda Y_\lambda$$

into the product $\prod_\lambda Y_\lambda$ as follows:

$$fx = \{f_\lambda x\} \in \prod_\lambda Y_\lambda.$$

The mapping obtained will be called the **product of the mappings** f_λ. If two mappings $f : X \to Y$ and $g : X \to Z$ are given, then we shall write $g < f$ if for every point $x \in X$ necessarily $g^{-1}gx \subseteq f^{-1}fx$.

§ 2. **θ_F-reconstruction**. Let X be a normal space, and let $\sigma = \{U_\alpha\}$ be its point-countable base. Suppose, moreover, that $F \subseteq X$ is a closed— a closed separable subspace. Consider the countable (by Lemma 2) system σ_F. We shall call a pair (U, U'), $U \in \sigma_F$, $U' \in \sigma_F$, **regular** if $[U] \subseteq U'$. Then, by the normality of the space X, there exists an open set V of type F_σ such that $[U] \subseteq V \subseteq V'$.

To each regular pair (U, U'), $U \in \sigma_F$, $U' \in \sigma_F$, we assign some one completely determined open set

$$V^F_{(U,U')}$$

of type F_σ:

$$[U] \subseteq V^F_{(U,U')} \subseteq U'.$$

The system of all $V^F_{(U,U')}$ thus constructed, over all regular pairs (U, U'), $U \in \sigma_F$, $U' \in \sigma_F$, will be denoted by θ_F. We shall call the indicated construction the **θ_F-rearrangement** of the system σ_F. It is not difficult to verify that θ_F is an at most countable system, forms an external base* of the set F, and all its elements intersect F.

Lemma 3. Let X be a finally compact p-space; $F \subseteq X$ a closed separable subspace; σ a point-countable base of X. For the system σ_F consider the system θ_F obtained from σ_F by a θ_F-rearrangement. Then there exists a metric space Y with a countable base and a perfect mapping $f : X \to Y$ such that

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* The theorem gives a partial answer to a problem of A. V. Arhangel' skii formulated at the end of this note (Problem B).

** A set $A \subseteq X$ is called marked under a mapping $f : X \to Y$ if $f^{-1}fA = A$.

*** By separability we mean the existence of a countable everywhere dense subset.

every $V^F_{(U,U')} \in \theta_F$ is distinguished under this mapping; moreover, under this mapping the set F will also be distinguished, and the restriction $f|F$ to F is a homeomorphism. In addition, as a consequence, F has type G_δ.

3. Proof of the main theorem 1. Let X be a finally compact p-space with a point-countable base $\sigma = \{U_\alpha\}$. Consider a metric space Y_1 with a countable base and a perfect mapping $f_1 : X \to Y_1$. Let $F_1 \subseteq X$ be a subset closed in X for which $f_1 F_1 = Y_1$, but $f_1 : F_1 \to Y_1$ is irreducible. Then the set F_1 is separable, and consequently for it the system σ_{F_1} is countable. Consider for the system σ_{F_1} the system θ_{F_1} obtained from σ_{F_1} by a θ_{F_1}-rearrangement. Then, by Lemma 3, there exists a metric space Y_2 with a countable base and a perfect mapping $f_2 : X \to Y_2$ such that every $V^{F_1}_{(U,U')} \in \theta_{F_1}$ is distinguished under the mapping f_2. Moreover, it may always be assumed that $f_2 < f_1$. Now consider such a closed set $F_2 \subseteq X$ that $f_2 F_2 = Y_2$ and that $f_2 : F_2 \to Y_2$ is irreducible. We obtain $F_1 \subseteq F_2$. Consider σ_{F_2} and the system $\theta_{F_2} = \{V^{F_2}_{(U,U')}\}$, obtained by a θ_{F_2}-rearrangement, and again apply Lemma 3 to the system θ_{F_2}, and so on. As a result we obtain perfect mappings

$$f_i : X \to Y_i, \qquad f_1 > f_2 > \cdots > f_i > \cdots,$$

onto metric spaces with a countable base, and closed separable sets

$$F_i \subseteq X, \qquad F_1 \subseteq F_2 \subseteq \cdots \subseteq F_i \subseteq \cdots.$$

The following conditions will then be fulfilled:

1°. $f_{iF}i = Y_i$ and $f_i : F_i \to Y_i$ is an irreducible perfect mapping.

2°. All elements $V^{F_{i-1}}_{(U,U')} \in \theta_{F_{i-1}}$ are distinguished under $f_i : X \to Y_i$. Here $\theta_{F_{i-1}}$ is the system obtained by a $\theta_{F_{i-1}}$-rearrangement of the system $\sigma_{F_{i-1}}$.

Since $\theta_{F_{i-1}}$ is an external base of the set F_{i-1}, the closed set F_{i-1} is distinguished under the mapping $f_i : X \to Y_i$, and f_i is a homeomorphism on the set F_{i-1}.

Now consider the set $\Phi = [\bigcup F_i]$ and the system $\theta = \bigcup_{i=1}^{\infty} \theta_{F_i}$. We shall prove that θ forms an external base of the closed set Φ. Indeed, let $x_0 \in \Phi$, and let Ox_0 be an arbitrary neighborhood of the point x_0 in X. Consider elements $U \in \sigma$ and $U' \in \sigma$ such that

$$x_0 \in U \subseteq [U] \subseteq U' \subseteq Ox_0.$$

Since $x_0 \in \left[\bigcup_{i=1}^{\infty} F_i\right]$, there exists an i_0 such that $U \cap F_{i_0} \neq \Lambda$

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[*] A system $\sigma = \{U\}$ of open sets of the space X is called an **external base of the set** $F \subseteq X$ if for any point $x_0 \in F$ and any of its neighborhoods Ox_0 in X there exists an element $U \in \sigma$ such that $x_0 \in U \subseteq Ox_0$ (see (8)).

and $U' \cap F_{i_0} \neq \Lambda$. Then $U \in \sigma_{F_{i_0}}$ and $U' \in \sigma_{F_{i_0}}$; consequently, (U, U') is a regular pair for the system

$$\sigma_{F_{i_0}} = \mathcal{E}(U \in \sigma,\ U \cap F_{i_0} \neq \Lambda).$$

Consider the system $\theta_{F_{i_0}}$, obtained by the $\theta_{F_{i_0}}$-refinement of the system $\sigma_{F_{i_0}}$. By the definition of a $\theta_{F_{i_0}}$-refinement, for the regular pair (U, U') there is an open set $V^{F_{i_0}}_{(U,U')} \in \theta_{F_{i_0}}$ of type F_σ for which

$$[U] \subseteq V^{F_{i_0}}_{(U,U')} \subseteq U'.$$

Then

$$x_0 \in U \subseteq [U] \subseteq V^{F_{i_0}}_{(U,U')} \subseteq U' \subseteq Ox_0.$$

Consequently,

$$\theta = \bigcup_{i=1}^{\infty} \theta_{F_i}$$

is an external base of the closed set

$$\Phi = \left[\bigcup_{i=1}^{\infty} F_i\right].$$

We shall prove that $\Phi = X$. Suppose the contrary; let

$$X \setminus \Phi \neq \Lambda.$$

Consider the mapping

$$f_i : X \to Y \subseteq \prod_{i=1}^{\infty} Y_i,$$

which is the product of the mappings $f_i : X \to Y_i$. The mapping f is also perfect (see (6)), and for every point $x \in X$ necessarily

$$f^{-1}fx = \bigcap_{i=1}^{\infty} f_i^{-1} f_i x.$$

Moreover, each $V \in \theta$ is marked under the mapping f (see (6), p. 108), but, since θ is an external base of the closed set Φ, Φ is also marked under the mapping f, i.e. $f^{-1}f\Phi = \Phi$, and $f|\Phi$ is a homeomorphism. Then the set $\Gamma = X \setminus \Phi$ will be marked, i.e., if $x_0 \in X \setminus \Phi$, then

$$f^{-1}fx_0 \subseteq X \setminus \Phi$$

or

$$f^{-1}fx_0 \cap \Phi = \Lambda.$$

On the other hand, $f_i : F_i \to Y_i$ is irreducible; consequently,

$$f_i^{-1} f_i x_0 \cap F_i \neq \Lambda$$

for every i, and hence, all the more,

$$f_i^{-1} f_i x_0 \cap \Phi \neq \Lambda.$$

Moreover,

$$\Phi_1 \supseteq \Phi_2 \supseteq \cdots \supseteq \Phi_i \supseteq \cdots,$$

since

$$f_1 > f_2 > \cdots > f_i > \cdots.$$

Consequently, the intersection of the bicompact sets Φ_i is nonempty:

$$\bigcap_{i=1}^{\infty} \Phi_i \neq \Lambda.$$

But then

$$f^{-1} f x_0 \cap \Phi = \bigcap_{i=1}^{\infty} (f_i^{-1} f_i x_0 \cap \Phi) = \bigcap_{i=1}^{\infty} \Phi_i \neq \Lambda$$

for $x_0 \in X \setminus \Phi$. We have obtained a contradiction. Thus necessarily $\Phi = X$, and therefore the mapping

$$f : X \to Y$$

is a homeomorphism into the metric space

$$\prod_{i=1}^{\infty} Y_i$$

with a countable base, and the base σ is countable.

The main theorem is completely proved.

§ 4. The general case

Theorem 2. Let a normal space X have pointwise weight$*$ τ and admit a perfect mapping onto a space of weight τ. Then the weight of the space X is equal to τ.

The proof of Theorem 2 is analogous to the proof of Theorem 1 and is based on the following lemmas.

Lemma 4. Let a normal space X admit a perfect mapping onto a space Y of weight τ, and let $\sigma = \{U\}$ be a system of open subsets of X of type F_σ and of cardinality $\leq \tau$. Then there exists a space Z of weight $\leq \tau$ and a perfect mapping

$$g : X \to Z,$$

under which all $U \in \sigma$ are marked.

Lemma 5. Suppose that in a normal space X there is a base σ of pointwise cardinality $\leq \tau$, and $F \subseteq X$ is a closed set in which there exists a dense set of cardinality $\leq \tau$. Then the system

$$\sigma_F = \mathcal{E}(U \in \sigma,\ U \cap F \neq \Lambda)$$

has cardinality $\leq \tau$.

Lemma 6. Let

$$f : X \to Y$$

be a perfect irreducible mapping of the space X onto a space Y of weight τ. Then the space X has a dense subset of cardinality τ.

$*$ A system σ of open subsets of X has pointwise cardinality $\leq \tau$ if any point $x \in X$ is contained in no more than τ elements $U \in \sigma$. The space X has pointwise weight τ if in X there exists a base σ of pointwise cardinality $\leq \tau$, but there does not exist a base of pointwise cardinality $\leq \tau' < \tau$.

What spaces, then, admit perfect mappings onto spaces of weight τ? In order to answer this question, let us introduce the following definitions.

Definition 1. A space X is called (τ, ∞)-compact if from every open cover ω of it one can extract a subcover of cardinality $\leq \tau$.

Definition 2. Let X be a completely regular space, and let bX be its bicompact extension. We shall call the space X a p^τ-space if in bX there exists a feathering (in the sense of A. V. Arhangel' skii (^{1,2})) of cardinality τ.

Theorem 3. *In order that a space X admit a perfect mapping onto a space of weight $\leq \tau$, it is necessary and sufficient that the space X be a (τ, ∞)-compact p^τ-space.*

§ 4. **Hereditarily finally compact spaces and statements of problems.** At the seminar of P. S. Aleksandrov the following problem was posed.

Problem 1. Will a hereditarily finally compact space X with a point-countable base σ be metrizable?

This problem, it seems to me, has transcendental difficulties, and we shall now show that it is, at least in one direction, connected with the classical Suslin problem.

Theorem 4. *Suppose there exists a nonseparable ordered bicompactum X satisfying the Suslin condition. Then there exists a nonmetrizable hereditarily finally compact space $X_0 \subseteq X$ with a point-countable base.*

Proof. Indeed, an ordered bicompactum satisfying the Suslin condition is perfectly normal, hence hereditarily finally compact and satisfies the first axiom of countability. To each transfinite number $\lambda < \omega_1$ assign a point $x_\lambda \in X$ in such a way that $x_\lambda \in X \setminus [\{x_{\lambda'}, \lambda' < \lambda\}]$. The points can be chosen in this way, since X is nonseparable. Now for each point x_λ consider a countable system of neighborhoods $\{U_i x_\lambda\} = \sigma_\lambda$, under the condition

$$[U_i x_\lambda] \subseteq X \setminus [\{x_{\lambda'}, \lambda' < \lambda\}],$$

forming a base at the point x_λ. Consider the subspace $X_0 = \{x_\lambda, \lambda < \omega_1\}$, the system $\sigma = \bigcup_\lambda \sigma_\lambda$, and the system $\sigma^* = \{U \cap X_0,\ U \in \sigma\}$. It is not hard to prove that σ^* is a point-countable base, and since the space X is nonseparable, it follows that the hereditarily finally compact space X_0 with the point-countable base σ^* is nonmetrizable.

Problem A. Is Problem 1 not equivalent to the Suslin problem?

Problem B. Does there exist a nonseparable perfectly normal bicompactum? Is this problem not equivalent to the Suslin problem?

Problem V. Will a paracompact p-space with a point-countable base be metrizable? (A. V. Arhangel' skii' s problem.)

In connection with A. V. Arhangel' skii' s problem we formulate a narrower problem.

Problem G. Suppose a space X admits a perfect irreducible mapping onto a metric space, i.e. it is coabsolute with some metric space and has a point-countable base. Will X be metrizable?

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Note: Figure translations are in progress. See original paper for figures.