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MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 517.544

MATHEMATICS

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A DISCONTINUOUS PROBLEM OF LINEAR CONJUGATION FOR ANALYTIC FUNCTIONS ON RIEMANN SURFACES

(Presented by Academician I. N. Vekua on 8 IV 1966)

Let on a closed Riemann surface R of genus ρ there be given a contour Γ, consisting of a finite number of mutually nonintersecting closed oriented curves satisfying the Lyapunov condition. Suppose, further, that measurable functions G and g are given on Γ. It is required to determine a function Φ, holomorphic in $R-\Gamma$, belonging to the class $H_p,\ p \geq 1^*$ on each component of $R-\Gamma$, whose boundary values almost everywhere on Γ satisfy the equality

$$\Phi^+ = G\Phi^- + g. \tag{1}$$

This problem has been well studied in the planar case under various restrictions on G and g (see, for example, ([2–5])). On a Riemann surface it was studied for G and g satisfying the Hölder condition ([6,7]) and when G has discontinuities of the first kind ([8,9]).

In the present note a method is given, analogous to the well-known alternating method of Schwarz, which makes it possible to investigate problem (1) on a Riemann surface under restrictions on G and g adopted in the papers ([2–5]).

Let on R there be given two domains K_0 and K_1, $\overline{K}_1 \subset K_0$, bounded by analytic curves and conformally equivalent to planar circular rings. Suppose, further, that in $K_0 - K_1$ a harmonic function u_0 is given.

Theorem 1. The condition

$$\int_{\partial K_0} du_0^* = 0 \tag{2}$$

is necessary and sufficient for the existence of a function u with the following properties:

A. u is single-valued and harmonic on $R - K_1$.

B. $u - u_0$ is the restriction to $K_0 - K_1$ of a function single-valued and harmonic in K_0.

The necessity of condition (2) is obvious.

Sufficiency. Consider sequences of harmonic functions $\{u_n\}$ and $\{v_n\}$, defined in the domains $R - K_1$ and K_0, respectively, by the boundary conditions

$$u_n = v_{n-1} + u_0 \quad \text{on } \partial(R - K_1), \tag{3}$$

$$v_n = u_n - u_0 \quad \text{on } \partial K_0 \tag{4}$$

$$(n = 1, 2, \dots;\ v_0 \equiv 0).$$

Map the domain K_0 by means of an analytic function τ one-to-one and conformally onto the planar annulus $1 < |z| < r$ and choose two numbers $r_1, r_2,\ 1 < r_1 < r_2 < r$, with $\ln r_1 = \ln r/r_2$, so that $\tau^{-1}(r_2 < |z| < r_1)$ and $\tau^{-1}(r_2 < |z| < r)$ lie in different components of $K_0 - K_1$. On the basis–

* For the definition of the Hardy classes H_p on finite Riemann surfaces, see ([1]).

of equalities (2), (3), and (4) it is easy to show that

$$\int_0^{2\pi} \left[v_n\left(e^{i\varphi}\right) - v_{n-1}\left(e^{i\varphi}\right) + v_n\left(re^{i\varphi}\right) - v_{n-1}\left(re^{i\varphi}\right)\right] d\varphi = 0. \tag{5}$$

Consequently,

$$\min_{K_0}(v_n - v_{n-1}) \le 0, \qquad \max_{K_0}(v - v_{n-1}) \ge 0.$$

Denote by $s(w, M)$ the oscillation of the function w on the set M. From the last inequalities it follows that

$$|v_n - v_{n-1}| \le s(v_n - v_{n-1}, K_0). \tag{6}$$

According to Lemma 3 of [10],

$$s(v_n - v_{n-1}, K_1) \le q\, s(v_n - v_{n-1}, K_0), \tag{7}$$

where $q < 1$ is a constant depending only on K_0 and K_1. From (3), (4), and (7) we obtain

$$s(v_{n+1} - v_n, K_0) \le q^n s(v_1, K_0). \tag{8}$$

The last inequality, together with inequality (6), entails the uniform convergence of the sequence $\{v_n\}$ in K_0. From the convergence of $\{v_n\}$ there follows the convergence of $\{u_n\}$ in $R-K_1$. Denote $v = \lim_{n\to\infty} v_n$, $u = \lim_{n\to\infty} u_n$. Passing to the limit in equalities (3) and (4), we obtain that the functions v and $u - u_0$ coincide in $K_0 - K_1$. The theorem is proved.

Let now a function $h \in L_p$, $p \geq 1$, be given on Γ. Consider the so-called Sokhotski problem: it is required to determine on $R - \Gamma$ a function $F \in H_p$ whose boundary values satisfy on Γ the condition

$$F^+ - F^- = h. \tag{9}$$

If a solution of the problem exists, then, as is easy to see, $\int_\Gamma h\,dw = 0$ for any differential of the first kind dw.

Choose for each Γ_j a doubly connected neighborhood K_j, map it by means of an analytic function τ_j onto a plane annulus, and construct in the plane a piecewise-holomorphic function $f_j \in H_p$ with respect to the boundary condition prescribed on $\gamma_j = \tau_j(\Gamma_j)$,

$$f_j^- - f_j^+ = h \circ \tau_j^{-1}. \tag{10}$$

The function $F_j = f_j \circ \tau_j$, defined in $K_j - \Gamma_j$, is holomorphic, belongs to H_p, and satisfies condition (9) on Γ_j. According to Theorem 1, there exists a function u_j, harmonic in $R - \Gamma_j$, such that $u_j - \operatorname{Re} F_j$ is the restriction to $K_j - \Gamma_j$ of a single-valued function harmonic in K_j. On each component of $R - \Gamma_j$ construct the function

$$F_j' = u_j + iu_j^*. \tag{11}$$

The restriction of any branch F_j' to $K_j - \Gamma_j$ belongs to H_p and satisfies on Γ_j the condition

$$F_j'^+ - F_j'^- = h + iC_j, \tag{12}$$

where C_j is a real constant depending on the chosen branch of F_j'.

Form the differential $dF = \sum_{j=1}^m dF_j$ and compute its periods on $R - \Gamma$. The calculation of the periods is carried out with the aid of the bilinear relation, taking (12) into account. The following assertion holds: for any cycle $l \subset R - \Gamma$ there exists a differential of the first kind dZ_l, depending only on l, such that

$$\int_l dF = \operatorname{Im} \int_\Gamma h\,dZ_l.$$

Let $\int_l dF = 0$ for any $l \subset R - \Gamma$. Then the analytic function

$$F = \int dF$$

is single-valued on $R - \Gamma$, belongs to H_p, and satisfies the boundary-condition

$$F^+ - F^- = h + iB, \tag{13}$$

where B is a real function, equal to a constant on each Γ_j. The value B on a nondividing component of the contour Γ can be made zero by adding a constant to F. The value B_j of the function B on a nondividing Γ_j can be computed. To this end one must take an Abelian differential of the first kind dZ_{Γ_j}, whose real part has a single nonzero period, equal to one along Γ_j.

From Cauchy' s theorem it follows that

$$B_j = \operatorname{Im} \int_\Gamma h \, dZ_{\Gamma_j}.$$

All that has been said confirms the validity of the following theorem.

Theorem 2. *For the solvability of the Sokhotski problem it is necessary and sufficient that*

$$\int_\Gamma h \, dw_\nu = 0 \qquad (\nu = 1, 2, \dots, \rho),$$

where $dw_1, dw_2, \dots, dw_\rho$ is a basis of the space of Abelian differentials of the first kind.

As mentioned above, the investigation of problem (1) is carried out under the assumptions adopted in papers ([2–5]). In the planar case it follows from these assumptions that:

I. $\ln G \in L_{p_1},\ p_1 \geq 1$.

II. $G = X^+(t)/X^-(t),\ t \in \Gamma$, where $X^\pm(z) \in H_p,\ 1/X^\pm(z) \in H_q,\ q = p/(p-1)$, and $X(z)$ has neither zeros nor poles outside Γ, except at the point at infinity, where the order of $X(z)$ is equal to the index of the problem.

III. The general solution of problem (1) is given by the formula $\Phi = XF + XP$, where F is a solution of the Sokhotski problem determined by the jump g/X^+, and P is a polynomial.

Thus, without formulating the restrictions imposed on G and g, we shall assume that the functions $G \circ \tau_j^{-1}$ and $g \circ \tau_j^{-1}$ satisfy conditions I-III on γ_j.

Let us first consider the homogeneous problem

$$\Phi^+ = G\Phi^-. \tag{14}$$

For the given G one can construct a function G_0, satisfying the Hölder condition on Γ, such that

$$\frac{1}{2\pi i}\int_{\Gamma_j} d\ln G_0 = \operatorname{ind}_{\gamma_j}(G\circ\tau_j^{-1}) \qquad (j=1,2,\dots,m),$$

$$\frac{1}{2\pi i}\int_{\Gamma} \ln G_0\, dw_\nu = \frac{1}{2\pi i}\int_{\Gamma} \ln G\, dw_\nu \qquad (\nu=1,2,\dots,\rho).$$

The functions $\dfrac{G}{G_0}\circ\tau_j^{-1}$ satisfy conditions I, II and

$$\operatorname{ind}_{\gamma_j}\left(\frac{G}{G_0}\circ\tau_j^{-1}\right) = 0.$$

Taking this into account, on the basis of Theorems 1 and 2 it is not difficult to establish that the homogeneous problem with coefficient G/G_0 is solvable. Its solution X_0 is single-valued and nowhere vanishes outside Γ. Dividing both sides of (14) by X_0, we arrive at the boundary-value problem defined by the boundary condition

$$\Phi_0^+ = G_0\Phi_0^-. \tag{15}$$

If (15) is solvable in the class H_p, then its solution is automatically continuously extendable to Γ. Problems (14) and (15) are equivalent. The general solution of problem (14) is given by the formula $\Phi = X_0\Phi_0$. The number of linearly independent solutions of problem (15), and hence of problem (14), depends on the numbers

$$\frac{1}{2\pi i}\int_{\Gamma} \ln G\, dw_\nu \qquad (\nu=1,2,\dots,\rho) \quad (\text{see } (^{11,12})).$$

To study the solvability of the nonhomogeneous problem, let us consider the conjugate problem associated with it, consisting in finding on $R-\Gamma$ a holomorphic differential $d\Psi \in H_q$, whose boundary values satisfy on Γ the condition

$$d\Psi^+ = \frac{1}{G}\, d\Psi^-. \tag{16}$$

Dividing (1) and multiplying (16) by X_0^+, we obtain two problems defined by the boundary conditions

$$\Phi_1^+ = G_0 \Phi_1^- + \frac{g}{X_0^+}, \tag{17}$$

$$d\Psi_1^+ = \frac{1}{G_0}\, d\Psi_1^-. \tag{18}$$

The solution of problem (17) is sought in the class H_p, and that of problem (18) in the class of differentials continuously extendable to Γ. Using assumption III on the functions G and g, one can show that problems (1) and (17) are equivalent and their solutions are related by the relation $\Phi = X_0 \Phi_1$. The equivalence of problems (16) and (18) and the validity of the relation $d\Psi_1 = X_0 d\Psi$ are obvious.

Theorem 3. For problem (1) to be solvable, it is necessary and sufficient that

$$\int_\Gamma g\, d\Psi^+ = 0 \tag{19}$$

for every solution $d\Psi$ of the conjugate problem.

Necessity follows from the Cauchy theorem.

Sufficiency. We rewrite condition (19) in the form

$$\int_\Gamma \frac{g}{X_0^+}\, d\Psi_1^+ = 0. \tag{20}$$

It is not difficult to show that on Γ one can construct a function g_1, satisfying the Hölder condition, such that the problem determined by the boundary condition

$$\Phi_2^+ = G_0 \Phi_2^- + g/X_0^+ + g_1,$$

will have a solution in the class H_p. From the necessity of condition (19) it follows that

$$\int_\Gamma \left(\frac{g}{X_0^+} + g_1 \right) d\Psi_1^+ = \int_\Gamma g_1 d\Psi_1^+ = 0. \tag{21}$$

But, as is known, from (21) there follows the existence of a function Φ_3, holomorphic on $R - \Gamma$ and continuously extendable to Γ, whose boundary values satisfy the condition

$$\Phi_3^+ = G_0 \Phi_3^- + g_1.$$

A direct verification shows that $\Phi = \Phi_2 - \Phi_3$ is a solution of problem (17). Thus problem (17), and consequently problem (1), is solvable.

Remark. All those facts of the theory of boundary-value problems on Riemann surfaces with Hölder coefficients to which we have referred in the present note can be established starting from Theorem 1.

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Received
2 IV 1966

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Note: Figure translations are in progress. See original paper for figures.