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ON SOME PROPERTIES OF CONSTRUCTIVE GENERALIZED FUNCTIONS

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1. The present note is a continuation of the author' s work (^{5}). In this note all special terms and notations not explained are understood in the same way as in (^{2}-^{5}). In (^{5}) the notion of a constructive generalized function was defined; it is a constructive analogue of the classical notion of a generalized function defined in (^{6}, ^{7}).

Let us introduce some further notions and notations: let Φ_1, Φ_2 be constructive generalized functions and let a, b be real duplexes, $a < b$. Take a fixed natural number k such that $k \geq \max(|a|\square|b|)$. We shall say that Φ_1 is equal to Φ_2 in the interval $a\nabla b$, if there exists a word P of type such that $L(P) < l^*$ and

$$\forall x(a < x < b \supset I(k\square l - m\square\{\Phi_1\}_k)(x) -$$

$$- I(k\square l - n\square\{\Phi_2\}_k)(x) = G(P)(x)),$$

where $m \rightleftarrows |\Phi_1|_k,\ n \rightleftarrows |\Phi_2|_k,\ l \rightleftarrows \max(m\square n)$, and G is the algorithm defined in Sec. 1 of (^{5}).

It is not difficult to prove that in the indicated definition, instead of k one may take any natural number greater than or equal to $\max(|a|\square|b|)$.

We shall say that Φ_1 is equal to Φ_2 in the interval $a\nabla\infty$, if there exists an algorithm $\mathfrak{P}$ of type ($\dot{\rightarrow}$) possessing the following properties:

1) $\forall k\ (k > a \supset !\mathfrak{P}(k))$;

2) $\forall k\ (k > a \supset L(\mathfrak{P}(k)) < l_k)$;

3) $\forall kx\ (a < x \leq k \supset I(k\square l_k - m_k\square\{\Phi_1\}_k)(x) -$

$$- I(k\square l_k - n_k\square\{\Phi_2\}_k)(x) = G(\mathfrak{P}(k))(x)),$$

where $m_k \rightleftarrows |\Phi_1|_k,\ n_k \rightleftarrows |\Phi_2|_k,\ l_k \rightleftarrows \max(m_k \square n_k)$.

In an analogous way one may define the notion that Φ_1 is equal to Φ_2 in the interval $-\infty\nabla a$.

Theorem 1. *There exists an algorithm that transforms every word of the form* $a\nabla b\square\Phi_1$*, where* a, b *are real duplexes,* $a < b$*, and* Φ_1 *is a constructive generalized function, into a word of the form* $n\square\mathfrak{m}_1$ *such that* Φ_1 *is equal to* $\mathfrak{D}(n\square R(\mathfrak{m}_1))$ *in the interval* $a\nabla b$*.*

In other words, in every finite interval every constructive generalized function is equal to a derivative of some order of an almost uniformly continuous function.

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* The notation $<$ is understood in the same way as in (^{5}).

We shall say that the segment $a\triangle b$ is a **carrier** of the constructive generalized function Φ_1, and shall write

$$(a\triangle b \text{ carries } \Phi_1),$$

if Φ_1 is equal to zero in the intervals $-\infty\nabla a$ and $b\nabla\infty$.

Let Φ_1 be a constructive generalized function. We shall say that Φ_1 is a **constructive generalized function with bounded carrier** if

$$\exists k\,(-k\triangle k \text{ carries } \Phi_1).$$

Theorem 2. *There is no algorithm that transforms each constructive generalized function* Φ_1 *with bounded carrier into a natural number* k *such that* $(-k\triangle k$ *carries* $\Phi_1)$*.*

Theorem 3. a) *Every constructive generalized function with bounded carrier is a constructive generalized function of finite order;* b) *there is no algorithm that transforms each constructive generalized function with bounded carrier* Φ_1 *into a natural number* n *such that* Φ_1 *is a constructive generalized function of order* $\leq n$*.*

2. Passing to the consideration of sequences of constructive generalized functions and of the limits of such sequences, we first introduce some notation: let $\mathfrak{m}_1$ be a word of type $\mathfrak{m}$, i.e. the complete code of an almost uniformly continuous function. We shall denote $\mathfrak{D}(n\square R(\mathfrak{m}_1))$ by $\mathfrak{m}_1^{(n)}$. If two constructive generalized functions Φ_1 and Φ_2 are equal in the interval $-k\nabla k$, we shall write $\Phi_1 \underset{k}{=} \Phi_2$.

Let λ_1 be an algorithm of type $(\mathfrak{n} \to \mathfrak{m})$ and X a word of type $\mathfrak{m}$. We shall write $(X \operatorname{pred}_{C_k} \lambda_1)$ if $\xi(k\square X)$ is the limit of the sequence defined by the algorithm $(\tilde{\xi}_k\square\circ\lambda_1)$ in the space C_k.

Let φ be an algorithm of type $(\mathfrak{n} \to \Phi)$, i.e. a sequence of constructive generalized functions. We shall say that the constructive generalized function Φ_1 is the **limit of the sequence φ in the interval $-k\nabla k$** if

$$\exists i\, \mathfrak{m}_1 \lambda_1 (\Phi_1 \underset{k}{=} \mathfrak{m}_1^{(i)} \;\&\; \forall_n^{\sim} (\varphi(n) \underset{k}{=} \lambda_1(n)^{(i)}) \;\&\; (\mathfrak{m}_1 \operatorname{pred}_{C_k} \lambda_1)),$$

where λ_1 is a variable for algorithms of type $(\mathfrak{n} \to \mathfrak{m})$.

We shall call Φ_1 the **limit of the sequence** φ, if for every k, Φ_1 is the limit of the sequence φ in the interval $-k\nabla k$.

Theorem 4. a) *Every constructive generalized function is the limit of a sequence of regular constructive generalized functions;* b) *every constructive generalized function is the limit of a sequence of constructive generalized functions with bounded carriers.*

It is not difficult to prove the following assertions:

1) If $\mathfrak{m}_1$ is the limit in the space $\mathcal{C}$ of a sequence λ of words of type $\mathfrak{m}$ (i.e. for every k, $\xi(k\square\mathfrak{m}_1)$ is the limit in C_k of the sequence defined by the algorithm $(\tilde{\xi}_k\square \circ \lambda))$, then $R(\mathfrak{m}_1)$ is the limit of a sequence of constructive generalized functions defined by the algorithm $(R \circ \lambda)$.
2) If Φ_1 is the limit of the sequence φ of constructive generalized functions, then for whatever natural number m, $\mathfrak{D}(m\square\Phi_1)$ is the limit of the sequence of constructive generalized functions defined by the algorithm $(\mathfrak{D}_m\square \circ \varphi)$.

Let a real duplex a_0 be given. Let k be a natural number, $\mathfrak{m}_1^k$ a word of type $\mathfrak{m}^k$, and λ an algorithm of type $(\mathfrak{d} \dot{\to} \mathfrak{m}^k)$, applicable to any-

to any duplex $\alpha \neq \alpha_0$. We shall say that $\mathfrak{m}_1^k$ is the **limit** of λ as $\alpha \to \alpha_0$ in the space C_k, if

$$\forall i\, \exists j\, \forall \alpha\, (-k \leq x \leq k \,\&\, 0 < |\alpha - \alpha_0| < 2^{-j} \supset |\mathfrak{m}_1^k(x) - \lambda(\alpha)(x)| < 2^{-i}).$$

Let $\mathfrak{m}_1$ be a word of type $\mathfrak{m}$, and let λ be an algorithm of type $(\mathfrak{d} \to \mathfrak{m})$, applicable to every duplex $\alpha \neq \alpha_0$. We shall say that $\mathfrak{m}_1$ is the **limit** of λ as $\alpha \to \alpha_0$ in the space C_k, and shall write

$$(\mathfrak{m}_1 \operatorname{pred}_{C_k(\alpha \to \alpha_0)} \lambda),$$

if $\xi(k\square\mathfrak{m}_1)$ is the limit of $(\tilde{\xi}_{k\square} \circ \lambda)$ as $\alpha \to \alpha_0$ in the space C_k.

Now let φ be an algorithm of type $(\mathfrak{d} \to \Phi)$, applicable to every duplex $\alpha \neq \alpha_0$. We shall say that the constructive generalized function Φ_1 is the **limit** of φ as $\alpha \to \alpha_0$, if for every natural number k the following holds:

$$\exists i j \mathfrak{m}_1 \lambda_1 (\Phi_1 = \mathfrak{m}_1^{(i)} \,\&\, \forall \alpha\, (0 < |\alpha - \alpha_0| < 2^{-j} \supset$$

$$\supset \varphi(\alpha) = \lambda_1(\alpha)^{(i)} \,\&\, (\mathfrak{m}_1 \operatorname{pred}_{C_k(\alpha \to \alpha_0)} \lambda_1)),$$

where λ_1 is a variable for algorithms of type $(\mathfrak{d} \to \mathfrak{m})$, applicable to any duplex $\alpha \neq \alpha_0$.

3. We define the operator of a linear change of variable for constructive generalized functions.

Let $\alpha \neq 0$ and β be real duplexes, and let $\Phi_1 \simeq X\tau Y\tau Z\tau T$ be a constructive generalized function. Construct an algorithm η of type $(\mathfrak{n} \to \mathfrak{n})$ such that

$$\forall k\, (\eta(k) > |\alpha| k + |\beta|).$$

Construct algorithms λ of type $(\mathfrak{n} \to \mathfrak{m})$, N of type $(\mathfrak{n} \to \mathfrak{n})$, Q of type $(\mathfrak{n}\mathfrak{n} \to \mathfrak{n})$, such that

$$\forall n x\, (\lambda(n)(x) \simeq \langle X \rangle (n)(\alpha x + \beta));$$

$$\forall k\, (N(k) \simeq \langle Y \rangle (\eta(k)));$$

$$\forall k m\, (Q(k \Box m) \simeq \langle T \rangle (\eta(k) \Box m)).$$

It is possible to construct an algorithm P of type $(\mathfrak{n}\mathfrak{n} \to \mathfrak{y})$ such that, for any k and n, $L(P(k \Box n)) < p$ $(p \simeq N(k))$, and for every x in $-k \triangle k$:

$$\mathfrak{S}(p \Box \lambda(n))(x) + G(P(k \Box n))(x) =$$

$$= \frac{1}{\alpha^p} \cdot \mathfrak{S}(p \Box \langle X \rangle (n)) + G(\langle Z \rangle (\eta(k) \Box n))(\alpha x + \beta).$$

It is not difficult to prove that the word $\xi \lambda 3 \tau \xi N 3 \tau \xi P 3 \tau \xi Q 3$ represents a constructive generalized function. One can construct an algorithm which transforms each word of the form $\alpha \Box \beta \Box \Phi_1$ into the corresponding word $\xi \lambda 3 \tau \xi N 3 \tau \xi P 3 \tau \xi Q 3$. We shall denote this algorithm by $\mathcal{L}$.

Let c be a real duplex. Denote by $\tilde{c}$ a word of type $\mathfrak{m}$ such that

$$\forall x\, (\tilde{c}(x) = c),$$

and then denote $R(\tilde{c})$ by $\bar{c}$. Then $\bar{c}$ is a constructive generalized function.

A constructive generalized function Φ_1 is called **constant** if there exists a real duplex c such that $\Phi_1 = \bar{c}$.

Let a constructive generalized function Φ_1 and a real duplex x_0 be given. We construct an algorithm φ of type $(\mathrm{D}\dot{\rightarrow}\Phi)$, such that for any $a \neq 0$:

$$\varphi(a) \simeq \mathcal{L}(a\Box x_0\Box\Phi_1). \tag{1}$$

Copying the classical proof (see (^{6}), § 16), we can prove the following assertion: if there exists such a constructive generalized function Φ_2 that Φ_2 is the limit of φ as $a \to 0$, then Φ_2 is constant.

We shall say that a real duplex c is a **value** of the constructive generalized function Φ_1 at the point x_0, and shall write

$$(c\ \underline{\mathrm{val}}_{x_0}\ \Phi_1),$$

if $\bar{c}$ is the limit of the algorithm φ, defined by formula (1), as $a \to 0$.

The point x_0 is called a **regular point** of the constructive generalized function Φ_1 if $\exists c(c\ \underline{\mathrm{val}}_{x_0}\ \Phi_1)$; it is called a **singular point** of Φ_1 if it is not a regular point of Φ_1.

It is easy to prove that if Φ_1 is equal to $R(\mathfrak{sh}_1)$ in some interval $a\nabla b$, where $\mathfrak{sh}_1$ is a word of type $\mathfrak{sh}$, then every point x_0 in $a\nabla b$ is a regular point of Φ_1, and whatever the point x_0 in $a\nabla b$, $(\mathfrak{sh}_1(x_0)\underline{\mathrm{val}}_{x_0}\Phi_1)$.

Łojasiewicz’ s theorem (Theorem 2.3 (^{8})) is transferred into constructive mathematics in the following way.

Let Φ_1 be a constructive generalized function and x_0 a real duplex. Let $k > |x_0|$. The real duplex c is a value of Φ_1 at the point x_0 if and only if there exist a number n and a word $\mathfrak{sh}_1$ such that Φ_1 is equal to $\mathfrak{sh}_1^{(n)}$ in $-k\nabla k$, and $c/n!$ is the limit of the function f, defined by the formula

$$\forall x\,(f(x) \simeq \mathfrak{sh}_1(x) : (x - x_0)^n),$$

as $x \to x_0$.

Theorem 5. *One can construct a constructive generalized function Φ_1 possessing the property: there is no function f of a real variable, defined at all regular points of Φ_1, and such that at every regular point x_0 of Φ_1 one has $(f(x_0)\underline{\mathrm{val}}_{x_0}\Phi_1)$.*

The proof of this theorem is based on Theorem 5.1 from (^{1}) and the lemma of § 1, Chapter III from (^{4}).

Theorem 6. *There exists such a constructive everywhere-defined function f of a real variable that there is no constructive generalized function Φ_1 satisfying the condition:*

$$\forall x\,(f(x)\underline{\mathrm{val}}_x\Phi_1).$$

The question of a constructive analogue of the concept of generalized function, based on the concept of a linear continuous functional in the space of finite infinitely differentiable functions, will be considered in another communication.

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Note: Figure translations are in progress. See original paper for figures.