SOLVABILITY OF THE CAUCHY PROBLEM FOR ABSTRACT PARABOLIC EQUATIONS WITH VARIABLE OPERATORS

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Abstract

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SOLVABILITY OF THE CAUCHY PROBLEM FOR ABSTRACT PARABOLIC EQUATIONS WITH VARIABLE OPERATORS

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The paper investigates the solvability of the following Cauchy problems in a Banach space E:

$$du(t)/dt + A(t)u(t) = f(t), \qquad u(0) = u_0, \tag{1}$$

$$d^2u(t)/dt^2 + A(t)du(t)/dt + B(t)u(t) = f(t), \qquad u(0) = u_0, \quad u'(0) = u_1. \tag{2}$$

By a solution of (1) ((2)) on $[0,T]$ we shall mean a continuous (continuously differentiable) on $[0,T]$, continuously differentiable (twice continuously differentiable) on $(0,T]$ function $u(t)$ satisfying, for each $t \in (0,T]$, equation (1) ((2)), the initial condition $u(0) = u_0$ (the initial conditions $u(0) = u_0,\ u'(0) = u_1$), and having, in addition, the property that the function $A(t)u(t)$ (the functions $A(t)du(t)/dt$ and $B(t)u(t)$) is continuous on $(0,T]$.

Problems (1) and (2) are studied under the assumption that the operator $A(t)$ satisfies the following conditions:

1°. The closed linear operator $A(t)$ $(t \in [0,T])$ has everywhere a dense domain of definition $D(A(t))$, and the inequality

$$\|R(\sigma + i\tau, -A(t))\| \leq C(\sigma + 1 + |\tau|^{\alpha})^{-1}$$

holds for all $\sigma \geq 0$ and some $\alpha \in ({}^5\!/_6, 1]$.

2°. For all $0 \leq \tau \leq t \leq T$ one has the inclusion $D(A(\tau)) \subset D(A(t))$; for arbitrary $0 \leq s \leq \tau,\ t \leq T$, the inequality

$$\|[A(t) - A(\tau)]A^{-1}(s)\| \leq C|t - \tau|^{5(1-\alpha)/\alpha + \varepsilon},$$

is satisfied, where ε is some number from $(0, (6\alpha - 5)/\alpha]$.

1. Solvability of problem (1).

For $\alpha = 1$, problem (1) has been well studied in works by both Soviet and foreign mathematicians. We note the papers (^{1}, ^{2}), where, in addition to 1° and 2°, it is assumed that $D(A(t))$ does not depend on t.

For $\alpha \in (5/6, 1]$, problem (1) was studied in paper (^{3}). In that paper it is also assumed that the domain of definition $D(A(t))$ is constant and that, for $\tau \geq 0$, the semigroups $e^{-\tau A(t)}$ generated by the operators $-A(t)$ are strongly continuous. We remark that the results concerning problem (1) overlap with the results of Ya. D. Mamedov and P. E. Sobolevskii (^{4}), where the strong continuity for $\tau \geq 0$ of the semigroups $e^{-\tau A(t)}$ and the constancy of the domain of definition $D(A(t))$ are also assumed.

Theorem 1. *Suppose that conditions 1° and 2° are fulfilled. Then there exists a unique evolution operator $U(t, \tau) \in B(E)$, defined for $0 \leq \tau < t \leq T$, with the following properties:*

a) *$U(t, \tau)$ is continuous for $0 \leq \tau < t \leq T$ and*

$$U(t, \tau) = U(t, s)U(s, \tau), \qquad \tau < s < t;$$

b) *the range of values of $U(t, \tau)$ belongs to $D(A(t))$, and the operator-function $A(t)U(t, \tau)$ is continuous for $0 \leq \tau < t \leq T$;*

c) *$U(t, \tau)$ is contin-*

is continuously differentiable with respect to t for $0 \leq \tau < t \leq T$, and

$$\partial U(t, \tau)/\partial t + A(t)U(t, \tau) = 0.$$

The evolution operator $U(t, \tau)$ constructed differs from the evolution operators constructed in other works ($^{1-6}$) in that it is not defined for $t = \tau$*. However, the following holds.

Theorem 2. *Let conditions 1° and 2° be satisfied. Then the operator-function $U(t, \tau)A^{-\delta}(\tau)$ $(\delta > (1 - \alpha)/\alpha)$ is strongly continuous in t, for $t \geq \tau$**.*

Theorem 3. *Let conditions 1° and 2° be satisfied. Suppose, moreover:*

3°. *For any $0 \leq \tau, t \leq T$ the inequality holds*

$$\|f(t) - f(\tau)\| \leq C|t - \tau|^{2(1-\alpha)/\alpha+\theta},$$

where θ is some number in $(0, (3\alpha - 2)/\alpha]$.

4°. *$u_0 \in D(A^\delta(0))$ for some $\delta > (1 - \alpha)/\alpha$.*

Then problem (1) has a solution $u(t)$, which is given by the formula

$$u(t) = U(t, 0)u_0 + \int_0^t U(t, \tau)f(\tau)\, d\tau. \tag{3}$$

Theorem 4. *Let conditions* $1^\circ, 2^\circ$, *and* 3° *be satisfied, with condition* 1° *holding for* $\alpha = 1$. *Suppose, moreover:*

5°. $u_0 \in E$ *is an arbitrary element.*

Then problem (1) has a solution $u(t)$, *which is given by formula (3).*

Linear parabolic equations of the second order are reduced to the problem

$$du(t)/dt + A(t)u(t) + B(t)u(t) = f(t), \qquad u(0) = u_0, \tag{4}$$

as a result of which the need arises to investigate it. Applying Theorems 3, 4 and the author' s results (^{7,8}), we establish the following theorems.

Theorem 5. *Let conditions* $1^\circ, 2^\circ, 3^\circ$, *and* 4° *be satisfied. Suppose, moreover:*

6°. *For some* $\gamma \in [0, \alpha)$ *the operator* $B(t)A^{-\gamma}(t)$ *is bounded and the inequalities hold*

$$\|B(t)A^{-\gamma}(t)\| \leq C \qquad (0 \leq t \leq T),$$

$$\|[B(t) - B(\tau)]A^{-1}(s)\| \leq C|t-\tau|^{5(1-\alpha)/\alpha+\varepsilon} \qquad (0 \leq s \leq \tau, t \leq T) *** .$$

Then problem (4) has a unique solution.

Theorem 6. *Let conditions* $1^\circ, 2^\circ, 3^\circ, 5^\circ$, *and* 6° *be satisfied, with condition* 1° *holding for* $\alpha = 1$.

Then problem (4) has a unique solution.

2. Solvability of problem (2)

In the author' s works (^{7,8}), for the investigation of a problem of type (2) with constant operators A and B, a method of reduction to a system was proposed. This method, also in the case of problem (2), leads to the desired goal. Here, for simplicity of exposition, we shall assume that the operator $B(t)A^{-1}(t)$ is bounded. However, analogous theorems can be formulated when the operator $B(t)A^{-1}(t)$ satisfies condition 1° and certain smoothness requirements.

Theorem 7. *Let conditions* 1° *and* 3° *be satisfied. Suppose, moreover:*

7°. $D(A(\tau)) \subset D(A(t))$ *for* $0 \leq \tau \leq t \leq T$; *the operator-function*

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[*] S. G. Krein informed us that his book on differential equations in Banach space contains an analogous theorem, obtained by him and O. I. Proz skaya, in which it is required that $\alpha \in (2/3, 1]$, but additional smoothness conditions are imposed on the operator $A(t)$.

[**] It seems to us that the operator-function $U(t,\tau)A^{(1-\alpha)/\alpha}(\tau)$ should be strongly continuous in t for $t \geq \tau$; however, we have not yet been able to prove this.

*** The boundedness of the operator $B(t)A^{-1}(s)$ for $0 \le s \le t \le T$ follows from general theorems on subordinacy (see, for example, (^{9})), since

$$B(t)A^{-1}(s) = B(t)A^{-\gamma}(t)A^{\gamma}(t)A^{-1}(s)$$

and the operator $A(t)A^{-1}(s)$ is bounded.

$A(t)A^{-1}(\tau)$ for $0 \le \tau \le t \le T$ is strongly continuously differentiable with respect to t*; the inequality holds

$$\|A'(t)A^{-1}(s) - A'(\tau)A^{-1}(s)\| \le C|t-\tau|^{5(1-\alpha)/\alpha+\varepsilon} \quad (0 \le s \le \tau,\ t \le T),$$

where ε is some number from $(0, (6\alpha-5)/\alpha]$.

8°. The operator $B(t)A^{-1}(t)$ $(t \in [0,T])$ is bounded, and the operator-valued function $B(t)A^{-1}(t)$ satisfies the inequality

$$\|B(t)A^{-1}(t) - B(\tau)A^{-1}(\tau)\| \le C|t-\tau|^{5(1-\alpha)/\alpha+\varepsilon} \quad (0 \le s \le \tau,\ t \le T).$$

9°. $u_0 \in D(A(0)) \cap D(B(0)), \quad u_1 \in D(A(0))$.

Then problem (2) has a unique solution.

In the case where the domain of definition $D(A(t))$ is constant, a stronger result holds:

Theorem 8. Suppose conditions $1^\circ, 3^\circ$, and 9° are fulfilled. Suppose, moreover,

10°. $D(A(t)) = D(A)$ does not depend on t; the operator-valued function $A(t)A^{-1}(0)$ satisfies the Hölder condition, i.e.

$$\|[A(t) - A(\tau)]A^{-1}(0)\| \le C|t-\tau|^{5(1-\alpha)/\alpha+\varepsilon} \quad (0 \le \tau,\ t \le T).$$

11°. The operator $B(t)A^{-1}(0)$ is bounded, and the inequality holds

$$\|[B(t) - B(\tau)]A^{-1}(0)\| \le C|t-\tau|^{5(1-\alpha)/\alpha+\varepsilon} \quad (0 \le \tau,\ t \le T).$$

Then problem (2) has a unique solution.

In the case where $D(A(t)) \equiv D(A)$ and $\alpha = 1$, Theorems 7 and 8 intersect with the results of P. E. Sobolevskii (^{10}).

3. In this section, using simple examples, we shall show that the range of applicability of the results obtained is much wider than that of those known previously.

Consider in the Hilbert space l_2 the self-adjoint operator $A(t) = \{(2-t)^n \delta_{nk}\}$ ($\delta_{nk} = 1$ for $n = k$; $\delta_{nk} = 0$ for $n \ne k$). The infinite system $du(t)/dt + A(t)u(t) = 0$ in l_2, or

$$du_n(t)/dt + (2-t)^n u_n(t) = 0 \quad (n = 1, 2, \dots, \infty) \tag{5}$$

is solved quite trivially. However, none of the abstract results listed by us $(^{1-6})$ can guarantee this, since the operator $A^{\alpha}(t)$, for any $0 < \alpha \leq 1$, has a variable domain of definition for $t \in [0,1]$. In the works $(^{3,11})$, conditions are imposed on $A(t)$ which do not require the constancy of $D(A^{\alpha}(t))$; however, replacing in (5) the function $2-t$ by a suitable function that is not differentiable but only Hölder, we obtain an example which certainly does not fall under that theory either. Nevertheless, it is easy to see that on the interval $[0,1]$ the operator $A(t)$ satisfies conditions 1° and 2°. Hence, by Theorem 1, there exists an evolution operator $U(t,\tau)$ generated by the operator $A(t)$. We note that condition 1° for the given operator is fulfilled with $\alpha = 1$. The following example shows that there exist operators satisfying condition 1° with $\alpha < 1$, and that the semigroups generated by them are not strongly continuous at zero. Consider the operator A in the space l_2, which is an infinite matrix with cells

$$\begin{pmatrix} ia_n + a_n^{\alpha} & 0 \\ a_n^{\beta} & ia_n + a_n^{\alpha} \end{pmatrix} \qquad (a_n \geq 1),$$

arranged along the main diagonal. If the numerical sequence a_n is unbounded, then the operator A is unbounded in l_2. Since the cells of the infinite matrix $[A+\sigma+i\tau]^{-1}$ have the form

$$\begin{pmatrix} [(\sigma+a_n^{\alpha})+i(\tau+a_n)]^{-1} & 0 \\ -a_n^{\beta}[(\sigma+a_n^{\alpha})+i(\tau+a_n)]^{-2} & [(\sigma+a_n^{\alpha})+i(\tau+a_n)]^{-1} \end{pmatrix},$$

then for $\alpha \leqslant \beta \leqslant 1$ and $\sigma \geqslant 0$ it is not hard to obtain the estimate

$$C_1(\sigma+|\tau|^{\alpha})^{-1} \leqslant \|[A+\sigma+i\tau]^{-1}\| \leqslant C_2(\sigma+|\tau|^{\alpha})^{-1}.$$

Thus, the operator $A + cI$ $(c > 0)$ satisfies condition 1°.

Finally, since the semigroup generated by the operator A has matrices of the form

$$\begin{pmatrix} e^{-(ia_n+a_n^{\alpha})t} & 0 \\ a_n^{\beta}te^{-(ia_n+a_n^{\alpha})t} & e^{-(ia_n+a_n^{\alpha})t} \end{pmatrix},$$

we have $C_1/t^{\beta/\alpha-1} \leqslant \|e^{-tA}\| \leqslant C_2/t^{\beta/\alpha-1}$. Thus, if $\alpha < \beta$, then the semigroup e^{-tA} is not strongly continuous at zero.

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* For $t = \tau$, the right derivative is meant.

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Note: Figure translations are in progress. See original paper for figures.

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