ON NON-SELF-INTERSECTING CURVES

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Abstract

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MATHEMATICS

V. I. PUPKO

ON NON-SELF-INTERSECTING CURVES ON CLOSED SURFACES*

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Theorem 1. Let an arbitrary closed surface $\mathfrak{F}$ be given with a curvilinear ray $\mathfrak{K}$, i.e., a non-self-intersecting curve which, when lifted to the universal covering surface, goes off to infinity. Then on the universal covering surface it has an asymptotic direction.

1. **Orientable surface of characteristic** $\chi = 2 - 2p < 0$**.** In this case the universal covering surface for $\mathfrak{F}$ is the Lobachevsky plane, and its development onto the plane is a (regular) $4p$-gon. The vertices of the $4p$-gons covering the plane form a lattice, each vertex (node) being a center of symmetry of the lattice of order $4p$, and the lattice (for any p) admits a rotation through 180°. It follows from this that any straight line L (half-line) passing through at least two nodes contains, along its entire length, nodes lying at equal distances from one another. Such straight lines (half-lines) will be called **rational**. Take any three consecutive nodes on L, denoting them by O_1, O_2, O_3. Join O_1 and O_2 by a broken line Λ_1, whose links are sides of the $4p$-gons. Construct a broken line Λ_2, symmetric to Λ_1 with respect to O_2 (Λ_2 joins O_2 and O_3). Finally, construct a broken line Λ_3, symmetric to Λ_2 with respect to O_3. The first link of Λ_1 forms with L the same angle as the first link of Λ_3, i.e., Λ_3 is obtained from Λ_1 by a shift along L. Continuing to construct broken lines in the same way, we find a node O_k such that the first link of the broken line Λ_k is equivalent to the first link of Λ_1. This means that the shift along the line L by the distance $d_L = |O_1O_k|$ belongs to the group of motions of the plane (the fundamental group of the surface $\mathfrak{F}$). Any two points on L lying at a distance that is a multiple of d_L will be called **corresponding with respect to** L. Let K be one of the preimages of the curve $\mathfrak{K}$ on the plane.

Lemma. The curve K cannot intersect any rational straight line L infinitely far in both directions.

Proof. Since K goes off to infinity, there exists a point on the curve after which

K lies entirely outside the circle of diameter 1 with center at a fixed point on L (the distance between corresponding points on L is taken to be 1). If now there were points of intersection of K with L arbitrarily far in both directions along L, one could choose two such points so that the part of the curve joining them also lay outside the indicated circle, and all its points of intersection with L lay outside a segment of length 1, both on one side and on the other side of this segment. But then there would be two consecutive (along K) points of intersection of the chosen part of K with L, x_1 and x_2, with distance greater than 1. Shifting the arc between x_1 and x_2 along L by 1, we obtain an intersection of this new arc with the original one, and hence a self-intersection of the curve $\mathfrak{K}$. The lemma is proved.

Proof of the theorem. Suppose the contrary assertion—

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* This problem was kindly suggested to me by D. V. Anosov.

of the theorem, we find an $\varepsilon > 0$ and a fixed angle Φ with vertex at an arbitrarily chosen node such that K, arbitrarily far away, intersects any ray lying inside this angle, as well as the sides of the angle. Draw inside Φ a rational ray γ, and take on γ a node of the lattice so far away that the straight line $\gamma^{\perp}$, drawn through this node perpendicular to γ, is parallel to, or diverges from, both one side and the other side of the angle Φ, i.e. lies entirely in the angle Φ. Then K must intersect $\gamma^{\perp}$ arbitrarily far away in both directions, which contradicts the lemma ($\gamma^{\perp}$ is rational).

2. Orientable surface, $\chi = 0$ (torus). In this case the universal covering for $\mathcal{F}$ is the Euclidean plane, and the lattice described above becomes the ordinary square lattice in the plane. We may assume that the beginning O of the curve K (one of the preimages of $\mathcal{K}$ on the Euclidean plane) coincides with a node of the lattice. Introduce a rectangular coordinate system in the plane with origin at the point O and axes running along the sides of the squares (the side length is taken to be 1). It is obvious that any straight line passing through the origin (or any ray issuing from the origin) with rational slope contains nodes of the lattice, i.e. is rational in the sense introduced earlier, and all nodes of such a straight line are corresponding with respect to it.

We divide the proof into two parts: a) K does not intersect arbitrarily far away any rational ray issuing from the origin; b) there exists at least one rational ray which K intersects arbitrarily far away.

a) Let γ_1 and γ_2 be two arbitrary rational rays. Starting from some point, K is entirely contained inside the angle $\gamma_1 O \gamma_2$ (one of the two complementary to 360°). Draw inside this angle a rational ray γ_3. Then, starting from some point, K lies entirely in one of the two resulting angles. Drawing successively rational rays $\gamma_4, \dots, \gamma_n, \dots$ in such a way that the nested angles obtained as a result of the division tend in magnitude to 0, we shall "squeeze" the curve into an ever smaller angle tending to 0. But this means the existence of an asymptotic direction for K.

b) Without loss of generality, the ray specified in the hypothesis may be taken to be the positive direction of the x-axis.

Lemma 1. *Under the hypotheses of the theorem, the curve K deviates boundedly from the x-axis at least on one side of it.*

Proof. Choose on the curve a point x_0 of intersection with the x-axis such that, starting from x_0, K no longer intersects the segment $[Ox_0]$. Let a be a point on the x-axis such that all points of intersection with the x-axis of the arc $\overset{\frown}{Ox_0}$ of the curve lie on the segment $[Oa]$. Take the point nearest to a on the right which is equivalent to x_0, and denote it by $x_0 + n$, considering n to be its distance from x_0. There exists a point x_1 of intersection of K with the x-axis, starting from which K no longer intersects the segment $[O, x_0 + n]$. Denote the arc $x_1\tilde{x}_1$, where $\tilde{x}_1$ is the point of intersection following x_1 along the curve, by α. Suppose, for definiteness, that α lies in the upper half-plane. Now shift the whole curve, starting from the point x_1, to the left by n, i.e. bring the point x_0+n into coincidence with x_0. We shall mark the points and arcs of the shifted curve (and the curve itself) with primes. Then K' will pass above the arc $O\tilde{x}_1$, and there will be on K' two consecutive points x_2' and $\tilde{x}_2'$ of intersection of K' with the x-axis, such that the arc $x_2'\tilde{x}_2'$ encloses the arc α from above. This means that the part $O\tilde{x}_2 \supset O\tilde{x}_1$ of the curve K is bounded below by the same constant as the arc $O\tilde{x}_1$. Starting from the point $\tilde{x}_2'$, the curve K' will already pass above the arc $O\tilde{x}_2$, and again there will be an arc $x_3'\tilde{x}_3' \subset K'$, enclosing the arc $x_2\tilde{x}_2$ from above. Thus the part $O\tilde{x}_3$ of the curve is already bounded. Continuing in this way to move along K', passing gradually through

through all points at distance n from one another, we see that the entire curve K is bounded below, which proves the lemma. (If the arc a were assumed to lie in the lower half-plane, we would arrive at boundedness of the curve from above.)

Proof of the theorem. Suppose that K has no asymptotic direction. Then there exists one more rational ray (issuing from the origin) which K intersects arbitrarily far out. But then any ray, in particular any rational ray, lying in the angle obtained (this angle is less than 180°), is intersected by the curve arbitrarily far out, and moreover the curve is not bounded on either side of this ray. We have obtained a contradiction with the lemma, which proves the theorem.

3. Nonorientable surfaces. Consider the orientable two-sheeted covering F_1 of the surface $\mathcal{F}$ and the inverse image K_1 of the curve $\mathcal{K}$ on it. Since the universal covering surface of $\mathcal{F}$ is at the same time also the universal covering surface of F_1, the inverse image of $\mathcal{K}$ and K_1 on the universal covering will be one and the same. Hence the validity of the theorem for orientable surfaces implies its validity also for nonorientable ones.

Theorem 2. *Under the hypotheses of Theorem 1, the curvilinear ray K deviates* *boundedly from any straight line of asymptotic direction.*

The proof of this theorem is more cumbersome and is based mainly on a lemma analogous to Lemma 1.

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Engineering and Construction Institute
named after V. V. Kuibyshev

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Note: Figure translations are in progress. See original paper for figures.