ON ONE POSSIBILITY OF STUDYING THE TRANSIENT REGIME FROM LINEAR TO NONLINEAR IN PARAMETRIC RESONANCE

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Abstract

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ON ONE POSSIBILITY OF STUDYING THE TRANSIENT REGIME FROM LINEAR TO NONLINEAR IN PARAMETRIC RESONANCE

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1. A broad class of problems in celestial mechanics, in the theory of the rigid body, in the theory of oscillations, in plasma physics, and also a number of problems of modern technology reduce to the investigation and solution of linear differential equations with periodic coefficients—equations of Hill type:

$$d^2w/dt^2 + [a - 2q\psi(t)]w = 0, \tag{1}$$

where $\psi(t)$ is a periodic function of t with period T; a and q are parameters. At the same time, the problem of investigating the stability of periodic solutions of nonlinear systems always also leads to Hill' s equation (1). If the modulation depth $h \equiv q/a \ll 1$, equation (1) can be studied with the aid of well-developed approximate methods ([2,3]); however, in the general case this is an exceptionally difficult problem. A distinctive feature of solutions of equation (1) is that, in the zones of instability, $w(t)$ increases without bound (even in the presence of friction, if it is linear), and the establishment of a stationary amplitude of oscillations in real phenomena approximately described by equation (1) is a consequence of the development of nonlinear effects, which may be of the most varied nature. An extensive literature, both theoretical and experimental, is devoted to these questions ([1,3–6]). We shall mention only the classical investigations of parametric resonance and of the influence of nonlinear effects on the establishment of a stationary amplitude of oscillations, which were carried out as early as the 1930s under the direction of L. I. Mandelstam and N. D. Papaleksi ([7]).

It is clear that in such problems the investigation, in real setups, of the transient regime in parametric resonance from linear oscillations described by equation (1) to nonlinear ones is of particular interest. The ideal solution of this problem would be a comparison of experimentally measured values of $w(t)$ in a setup approximately described by equation (1) with the exact solution $w(t)$ of this equation. However, as noted above, in the general case it is practically impossible to obtain an analytic solution of equation (1) for $h \gtrsim 1$, and its numerical integration up to $t = t_0$ (t_0 characterizes the moment of onset of the transient regime) is likewise a laborious problem.

It turns out that one can propose a simple experimental method for studying the transient regime, and its application becomes the more effective the stronger the inequality $t_0 \gg T$, i.e., the more slowly nonlinear effects manifest themselves in the real physical system. However, we shall impose no restrictions on the modulation depth $h = q/a$.

2. According to Floquet' s theorem ([2]), the general solution of equation (1), provided only that $\mu_1 \neq \mu_2$, can be written in the form:

$$w(t) = A_1 e^{\mu_1 t} f_1(t) + A_2 e^{\mu_2 t} f_2(t), \tag{2}$$

where $f_1(t)$ and $f_2(t)$ are periodic functions of t with period T, while μ_1 and μ_2 are characteristic exponents; A_1 and A_2 are arbitrary constants. Owing to the periodicity of the functions $f_1(t)$ and $f_2(t)$, we shall obviously obtain

$$w(t+T) = A_1 e^{\mu_1 t} \rho_1 f_1(t) + A_2 e^{\mu_2 t} \rho_2 f_2(t), \tag{3}$$

$$w(t+2T) = A_1 e^{\mu_1 t} \rho_1^2 f_1(t) + A_2 e^{\mu_2 t} \rho_2^2 f_2(t), \tag{4}$$

where the notation $\rho_1 \equiv e^{\mu_1 T}$ and $\rho_2 \equiv e^{\mu_2 T}$ has been introduced.

By simple algebraic transformations using expressions (2)–(4), it is easy to verify that, for fixed values of the parameters a and q, ρ_1 and ρ_2 must satisfy the equation

$$\rho_1 \rho_2 w(t) - [\rho_1 + \rho_2] w(t+T) + w(t+2T) = 0. \tag{5}$$

Usually, if $e^{\mu t} f(t)$ is a particular solution of Hill' s equation (1), then $e^{-\mu t} f(-t)$ is taken as the second linearly independent solution, and then, for $\mu \neq 0$, from equation (5) we obtain

$$\rho^2 - \rho \left[\frac{w(t) + w(t+2T)}{w(t+T)} \right] + 1 = 0. \tag{6}$$

Here $\rho \equiv e^{\mu T}$. Since the characteristic exponent depends only on the constant coefficients a and q of equation (1) and does not depend on t, the coefficient in the quadratic equation (6) likewise will not depend on the particular value of t; i.e., to determine ρ and, consequently, μ ($\mu \neq 0$), we obtain the equation

$$\rho^2 - \rho \left[\frac{w(0) + w(2T)}{w(T)} \right] + 1 = 0. \tag{7}$$

The values $w(t)$ at the points $t = T$ and $t = 2T$ are found by numerical integration of Hill' s equation (1), whereas the value $w(0)$ may be assigned arbitrarily, provided only that $w(0)$ and $dw/dt|_{t=0}$ do not vanish simultaneously. In practice it turns out that $w(0)$ should be chosen small in order to avoid large values of $w(t)$ as t increases, when $h \gg 1$. It is known in the literature ([2,8]) that the characteristic equation for ρ, related to (7), is as follows. Namely, if $u_1(t)$ and $u_2(t)$ are two particular solutions of Hill' s equation (1), satisfying the initial conditions:

$$u_1(0) = 1; \qquad u_1'(0) = 0;$$

$$u_2(0) = 0; \qquad u_2'(0) = 1,$$

then one can obtain

$$\rho^2 - [u_1(T) + u_2'(T)]\rho + 1 = 0.$$

However, the use of equation (7), which involves the values of only one function, moreover one not tied to strict initial conditions, appears methodologically more convenient for finding the characteristic exponents μ.

3. Using the definition $\rho \equiv e^{\mu T}$, equation (6) can be rewritten in the form:

$$w(t) = (e^{\mu T} + e^{-\mu T}) w(t + T) - w(t + 2T), \tag{8}$$

where by $w(t)$, $w(t+T)$, and $w(t+2T)$ we shall now understand experimentally measured quantities on the apparatus, approximately described by equation (1). The constant coefficient $\alpha \equiv e^{\mu T} + e^{-\mu T}$, appearing in equation (8), for a given apparatus with known values of a, q, and $\psi(t)$, and compatible with the solution of equations (1) and (7), can be calculated with arbitrarily high accuracy and is an unambiguous characteristic of the chosen theoretical model in describing the phenomenon under study. Beginning from some instant $t = t_0$, when nonlinear effects begin to exert a noticeable influence on the system, equation (8) will

will already be approximate in character and, by comparing the experimental values $w(t)$ with the corresponding values $w(t)$ from equation (8), it will be possible to judge the nature of the transient process.

If the physical phenomenon is described by a special case of Hill' s equation, namely by Mathieu' s equation, then for the computation of α one may use tables that we are preparing for publication.

We note that the method set forth above for finding the characteristic exponents μ for a linear second-order differential equation with periodic coefficients can be used to construct stability diagrams for arbitrary modulation depths h (see, for example, (2)).

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Note: Figure translations are in progress. See original paper for figures.