ON DEGENERATING ELLIPTIC EQUATIONS OF SECOND ORDER IN ARBITRARY SMOOTH DOMAINS

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Abstract

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MATHEMATICS

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ON DEGENERATING ELLIPTIC EQUATIONS OF SECOND ORDER IN ARBITRARY SMOOTH DOMAINS

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Here the results of papers ([1,2]) are extended to the case of a domain with an arbitrary smooth boundary.

Let the differential expression $Lu \equiv a^{ij}u_{x_i x_j} + a^i u_{x_i} + au$ be elliptic everywhere in the bounded domain $D \subset R_n$ with boundary $\dot{D}$, except for a closed set $D_0 \subset \dot{D}$. The coefficients $a^{ij}(x)$, $a^i(x)$, $a(x)$ are real functions belonging to C^1 in any closed subdomain $\bar{D} \setminus D_0$. In addition, we shall assume that $L\varphi - a\varphi$, for any function $\varphi \in C^\infty(R_n)$, is bounded above by a constant $c = c(\varphi)$.

Let the domain D belong to the class $A^{(3)}$ (see ([3])) and let $D_0 = \bar{D}' \cup \bar{D}''$, where $\bar{D}'$ and $\bar{D}''$ have no common points, and their boundaries Γ' and Γ'' on $\dot{D}$ belong to $A^{(2)}$. As is known, for sufficiently small $\delta > 0$ there exists a finite covering of the 2δ-neighborhood of $\dot{D}$ such that the intersection with D of each of the domains of the covering, by means of a nonsingular (for each domain of the covering, generally speaking, its own) transformation

$$y_m = y_m(x_1, x_2, \dots, x_n), \qquad m = 1, 2, \dots, n, \tag{1}$$

is transformed into a domain of special form (for example, a half-ball of radius $\rho \le 1$); moreover, the quantities $y_n(x)$ corresponding to different domains of the covering coincide on the intersection of these domains, and the quantities $y_{n-1}(x)$, in addition, coincide on the intersection of domains covering the 2δ-neighborhoods of Γ' and Γ''. Depending on this, we shall denote by: T'_α $(\alpha \in \mathfrak{A}')$ the domains of the covering having common points only with D'; T''_α $(\alpha \in \mathfrak{A}'')$ those having common points only with D''; E'_k $(k \in K')$ those having common points with Γ'; E''_k $(k \in K'')$ those having common points with Γ''. By G_β $(\beta \in \mathfrak{B})$ we shall denote the domains forming a finite covering of the part of $\bar{D}$ remaining after removing the 2δ-neighborhood of D_0. It may be assumed that the distances of $\bigcup_\alpha T'_\alpha$ to Γ', of $\bigcup_\alpha T''_\alpha$ to Γ'', and of $\bigcup_\beta G_\beta$ to D_0 are not less than δ. We shall denote by

$$\widetilde{L}v = b^{ml}v_{y_m y_l} + b^m v_{y_m} + av$$

the differential expression into which L is transformed after the transformation (1).

1. Consider the boundary-value problem:

$$Lu = f \text{ in } D; \qquad u = 0 \text{ on } D_1 = \dot{D} \setminus D_0; \qquad u = 0 \text{ on } \bar{D}''. \tag{2}$$

It follows from ([2]) that a solution of problem (2) exists and is unique if in $\bar{D} \setminus D_0$ there exist a homogeneous majorant $H_0(x)$ and an inhomogeneous majorant $H(x)$ of the equation $Lu = f$, possessing certain properties. The following conditions ensure the existence of such majorants.

1.1. For every $\alpha \in \mathfrak{A}'$ there exist functions $p_\alpha(s)$, $q_\alpha(s)$, continuous on $(0, 1]$, such that

$$b^n(y)/b^{nn}(y) \geq p_\alpha(y_n);$$

$$\int_0^1 e^{P_\alpha(s)}\,ds = \infty, \quad \text{where } P_\alpha(s) = \int_s^1 p_\alpha(t)\,dt; \quad \int_0^1 e^{-P_\alpha(r)} q_\alpha(r)\,dr < \infty;$$

$$\eta_n(x) \equiv \max_{1 \leq i \leq n} a^{ij}(x)(y_n)_{x_j} \leq c_\alpha e^{-P_\alpha(y_n(x))} \int_{y_n(x)}^1 e^{P_\alpha(s)}\,ds,$$

$$|f(x)| \leq b^{nn}(y(x)) q_\alpha(y_n(x)),$$

where $x \in T'_\alpha \cap D$ and $c_\alpha > 0$ is a constant.

1.2. For each $\alpha \in \mathfrak{A}''$ there exist functions $p_\alpha(s)$, $q_\alpha(s)$, continuous on $(0, 1]$, such that

$$b^n(y)/b^{nn}(y) \leq p_\alpha(y_n);$$

$$\int_0^1 e^{P_\alpha(s)}\,ds < \infty, \quad \text{where } P_\alpha(s) = \int_s^1 p_\alpha(t)\,dt;$$

$$\int_0^1 \int_s^1 e^{P_\alpha(s) - P_\alpha(r)} q_\alpha(r)\,dr\,ds < \infty, \quad \eta_n(x) \leq c_\alpha e^{-P_\alpha(y_n(x))} \int_0^{y_n(x)} e^{P_\alpha(s)}\,ds,$$

$$|f(x)| \leq b^{nn}(y(x))q_\alpha(y_n(x)),$$

where $x \in T''_\alpha \cap D$ and $c_\alpha > 0$ is a constant.

1.3. For each $k \in K'$ there exist functions $p_k(\sigma, t)$ and $q_k(t)$, continuous for $t \in [0,1]$ and $\sigma \in [-2,2]$, except for the set $t = 0$, $0 \leq \sigma \leq 2$, and moreover

$$\frac{b^n(y)}{b^{nn}(y)} \geq \max_{-2 \leq \tau \leq y_{n-1}} \int_{-2}^{\tau} p_k(\sigma, y_n)\, d\sigma;$$

$$-\frac{b^{n-1}(y)}{b^{n-1\, n-1}(y)} \geq \max_{y_n \leq s \leq 1} \int_s^1 p_k(y_{n-1}, t)\, dt;$$

$$-\frac{b^{n-1}(y)}{b^{n-1\, n-1}(y)} \geq \sup_{\substack{0<r<s \\ y_n<s<1}} \left\{ -\int_r^s p_k(y_{n-1}, t)\, dt \right\}.$$

Denote

$$Q_k(\tau, s) = \int_{-2}^{\tau} \int_s^1 p_k(\sigma, t)\, dt\, d\sigma$$

and suppose that

$$\overline{\lim}_{s\to 0} \sup_{-2<\tau<y_{n-1}<2} \left\{ -\int_{\tau}^{y_{n-1}} \int_s^1 p_k(\sigma, t)\, dt\, d\sigma \right\} < \infty;$$

$$\int_0^1 \int_{-2}^0 e^{Q_k(\tau,s)}\, d\tau\, ds = \infty; \quad \int_0^1 \int_{-2}^{\sigma} e^{Q_k(\tau,s)}\, d\tau\, ds < \infty \quad \text{for } \sigma < 0;$$

for $\sigma^* \in [0,2]$, $\sigma < \sigma^*$, $s > 0$ $(s \geq 0,\ \text{if } \sigma^* = 0)$

$$\overline{\lim}_{\rho\to 0} \int_{-2}^{\sigma^*} e^{Q_k(\tau,s)}\, d\tau \Big/ \int_{-2}^{\sigma} e^{Q_k(\tau,s)}\, d\tau \leq c_k, \quad \rho = \sqrt{(\sigma - \sigma^*)^2 + s^2};$$

the integral

$$\int_0^1 e^{-Q_k(\tau,r)} q_k(r)\, dr$$

converges uniformly for $\tau \in [-2, 2]$;

$$\Theta_n(x) \le c_k \min_{-2\le\tau\le y_{n-1}(x)} e^{-Q_k(\tau,y_n(x))} \int_{y_n(x)}^{1} e^{Q_k(\tau,s)}\,ds;$$

$$\Theta_{n-1}(x) \le c_k \min_{y_n(x)\le s\le 1} e^{-Q_k(y_{n-1}(x),s)} \int_{-2}^{y_{n-1}(x)} e^{Q_k(\tau,s)}\,d\tau;$$

$$|f(x)| \le b^{nn}(y(x))q_k(y_n(x)),$$

where

$$\Theta_i = \max\{\eta_i(x), \sqrt{\eta_i(x)}\} \quad (i = n-1, n);$$

$$\eta_{n-1}(x) \equiv \max_{1\le i\le n} a^{ij}(x)(y_{n-1})_{x_j}, \quad x \in E_k' \cap D,$$

$c_k > 0$ is a constant.

1.4. For each $k \in K''$ there exist functions $p_k(\sigma,t)$ and $q_k(t)$, continuous for all $t \in [0,1]$ and $\sigma \in [-2,2]$, except, possibly, for the set $t = 0,\ 0 \le \sigma \le 2$, and moreover

$$\frac{b^n(y)}{b^{nn}(y)} \le \min_{y_{n-1}\le\tau\le 2} \int_{-2}^{\tau} p_k(\sigma,y_n)\,d\sigma; \quad \frac{b^{n-1}(y)}{b^{n-1\,n-1}(y)} \le \inf_{\substack{s<r<1\\0<s<y_n}} \int_s^r p_k(y_{n-1},t)\,dt,$$

and such a function $\omega_k(\sigma)$, continuous for $\sigma \in [-2,2]$, equal to zero for $\sigma \in [0,2]$ and positive for $\sigma \in [-2,0)$, that the condition

$$\omega_k(y_{n-1}) \le \vartheta_k(y_{n-1}) \int_{y_{n-1}}^{0} \omega_k(\sigma)\,d\sigma,$$

where

$$\vartheta(y_{n-1}) \le b^{n-1}(y)/b^{n-1\,n-1}(y)$$

and $\vartheta(y_{n-1})$ is a bounded function for $y_{n-1} \in [-2,0]$, is satisfied.

We shall assume that

$$\overline{\lim_{s\to 0}} \sup_{-2<y_{n-1}<\tau<2} \int_{y_{n-1}}^{\tau} \int_s^1 p_k(\sigma,t)\,dt\,d\sigma < \infty; \tag{3}$$

$$\int_0^1 \int_{-2}^2 \int_s^1 e^{Q_k(\tau,s)-Q_k(\tau,r)} q_k(r)\,dr\,d\tau\,ds < \infty; \quad \sup_{-2<\tau<0} \int_0^1 e^{-Q_k(\tau,r)} q_k(r)\,dr < \infty;$$

$$\Theta_n(x) \le c_k \min_{y_{n-1}(x)\le\tau\le 2} e^{-Q_k(\tau,y_n(x))} \int_0^{y_n(x)} e^{Q_k(\tau,s)}\,ds +$$

$$+ c_k e^{-Q_k(y_{n-1}(x), y_n(x))} \int_{y_{n-1}(x)}^{0} \int_{s}^{0} \omega_k(\sigma)\, d\sigma;$$

$$\Theta_{n-1}(x) \le c_k \inf_{0<s<y_n(x)} e^{-Q_k(y_{n-1}(x), s)} \int_{y_{n-1}(x)}^{2} e^{Q_k(\tau, s)}\, d\tau;$$

$$\Theta_{n-1}(x) \le c'_k \quad \text{for } y_{n-1}(x) < 0; \qquad |f(x)| \le b^{nn}(y(x)) q_k(y_n(x)),$$

where $x \in E''_k \cap D$ and $c_k, c'_k > 0$ are constants. Note that from condition (3) there follows the existence of the integral

$$\int_0^1 \int_{-2}^{2} e^{Q_k(\tau, s)}\, d\tau\, ds.$$

1.5. $f(x)$ belongs to C^α $(0 < \alpha < 1)$ in any closed subdomain $\overline{D} \setminus D_0$, and there exist constants q_β $(\beta \in \mathfrak{B})$ such that

$$|f(x)| \le |a(x)| q_\beta$$

for $x \in G_\beta \cap D$.

1.6. The functions $p_\alpha(s), q_\alpha(s), Q_k(\tau, s), q_k(s)$ satisfy the following "compatibility" conditions:

$$\overline{\lim_{s\to 0}} |p_{\alpha_1}(s) - p_{\alpha_2}(s)| < \infty; \qquad \frac{1}{c_{\alpha_1\alpha_2}} q_{\alpha_2}(s) \le q_{\alpha_1}(s) \le c_{\alpha_1\alpha_2} q_{\alpha_2}(s),$$

where $0 < s < 1$, $T'_{\alpha_1} \cap T'_{\alpha_2} \ne \emptyset$ $(T''_{\alpha_1} \cap T''_{\alpha_2} \ne \emptyset)$ and $c_{\alpha_1\alpha_2} > 0$ is a constant;

$$\frac{1}{c'_{k\alpha}} e^{P_\alpha(s) - P_\alpha(r)} \le \int_{-2}^{y_{n-1}} e^{Q_k(\tau, s) - Q_k(\tau, r)}\, d\tau \le c'_{k\alpha} e^{P_\alpha(s) - P_\alpha(r)};$$

$$\frac{1}{c'_{k\alpha}} q_\alpha(s) \le q_k(s) \le c'_{k\alpha} q_\alpha(s),$$

where $0 < s < r \leqslant 1$; $\frac{1}{2} < y_{n-1} < 1$; $k \in K'$, $\alpha \in \mathfrak{A}'$; $T'_\alpha \cap E'_k \ne \emptyset$ and $c'_{k\alpha} > 0$ is a constant;

$$\frac{1}{c''_{k\alpha}} e^{P_\alpha(s) - P_\alpha(r)} \leqslant \int_{y_{n-1}}^{2} e^{Q_k(\tau, s) - Q_k(\tau, r)}\, d\tau \leqslant c''_{k\alpha} e^{P_\alpha(s) - P_\alpha(r)};$$

$$\frac{1}{c''_{k\alpha}} q_\alpha(s) \leqslant q_k(s) \leqslant c''_{k\alpha} q_\alpha(s),$$

where $0 < s < r \leqslant 1;\ \frac{1}{2} < y_{n-1} < 1;\ k \in K'',\ \alpha \in \mathfrak{A}'',\ T''_\alpha \cap E''_k \neq \emptyset$ and $c''_{k\alpha} > 0$ is a constant;

$$\lim_{s\to 0} \sup_{-1<\tau<2} \left| {}^*Q_{k_1}(\tau, s) - Q_{k_2}(\tau, s) \right| < \infty; \qquad \frac{1}{c_{k_1k_2}} q_{k_2}(s) \leqslant q_{k_1}(s) \leqslant c_{k_1k_2} q_{k_2}(s);$$

$$\frac{1}{c_{k_1k_2}} \omega_{k_2}(y_{n-1}) \leqslant \omega_{k_1}(y_{n-1}) \leqslant c_{k_1k_2} \omega_{k_2}(y_{n-1}),$$

where

$$0 < s < 1; \quad -2 < y_{n-1} < 0; \quad E'_{k_1} \cap E'_{k_2} \neq \emptyset \ (E''_{k_1} \cap E''_{k_2} \neq \emptyset) \quad \text{and} \quad c_{k_1k_2} > 0$$

is a constant.

Theorem 1. Suppose conditions 1.1–1.6 are satisfied. Then, for $a(x) \leqslant -a_0$, where a_0 is a sufficiently large positive constant, there exists a majorant $H(x)$ of the equation $Lu = f$ in $\bar{D} \setminus D_0$ such that $H(x) \geqslant 0$ in $\bar{D}$ and $H(x) \to 0$ as $x \to \bar{D}''$.

Theorem 2. Suppose the conditions of Theorem 1 are satisfied. Then there exists a majorant $H_0(x)$ of the equation $Lu = 0$, moreover $H_0(x) \geqslant 0$ in $\bar{D}$, and for every $\varepsilon > 0$ there exists $\rho = \rho(\varepsilon)$ such that $H(x)/H_0(x) \leqslant \varepsilon$ for x belonging to the ρ-neighborhood of D_0.

Theorem 3. Suppose the conditions of Theorem 1 are satisfied. Then there exists a unique solution of problem (2) satisfying the estimate $|u(x)| \leqslant H(x)$ $(x \in D)$.

2. Consider the boundary-value problem:

$$Lu = f \text{ in } D; \qquad Ru = 0 \text{ on } D_1 = \bar{D} \setminus D_0; \qquad u = 0 \text{ on } \bar{D}'', \tag{4}$$

where the boundary operator $R = -A\,\partial/\partial\gamma + B$ satisfies the conditions:

2.1. $A(x)$ and $B(x)$ are positive and belong to $C^{1+\alpha}$ in any closed subdomain of D_1.

2.2. The direction γ forms an acute angle with the inner normal to D_1, and in a δ-neighborhood $\Gamma'[\Gamma'']$ the inequalities hold

$$-A(x)\partial y_n/\partial\gamma \leqslant c\eta_n(x)\,[A(x)\partial y_n/\partial\gamma \leqslant c\eta_n(x)];$$

$$A(x)\partial y_{n-1}/\partial\gamma \leqslant c\eta_{n-1}(x)\,[-A(x)\partial y_{n-1}/\partial\gamma \leqslant c\eta_{n-1}(x)].$$

Theorems analogous to Theorems 1 and 2 are valid, from which the following follows.

Theorem 4. Suppose the conditions of Theorem 1, 2.1, 2.2 are satisfied and $B(x) \geqslant b_0$, where b_0 is a sufficiently large positive constant. Then there exists a unique solution of problem (4), satisfying the estimate

$$|u(x)| \leqslant H(x) \quad (x \in D).$$

3. The consideration of problems (2) and (4) with inhomogeneous boundary conditions is connected only with questions of extending functions given on the boundary into the interior of the domain in a certain class of functions. Therefore the corresponding results of work (2) carry over almost without change to the case of an arbitrary smooth domain.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.