On the approximation of linear differential equations with lag

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Abstract

The problem of approximating, on a finite time interval, the solutions $x(t)$ of an n-dimensional system of linear differential equations

$$dx(t)/dt = \sum_{i=1}^{n} A_i(t)x_i(t - h_i(t)) \tag{1}$$

with variable delays $0 \leq h_i(t) \leq h$ is considered. It is shown that the system (1) can be associated with a system of linear equations without delay, the solutions of which converge to the solution of the original system (1). The specified convergence is uniform in time t and over all initial functions $\varphi(\vartheta) = \{\phi_1, \dots, \phi_n\}$ with norm

$$\|\varphi(\vartheta)\| = \left(\sum_{j=1}^{n} \varphi_j^2(0) + \sum_{j=1}^{n} \int_{-h}^{0} \varphi_j^2(\vartheta)\, d\vartheta \right)^{1/2} \leq 1.$$

Bibliography: 5 items.

Full Text

Preamble

In this section, we consider the approximation of systems with time-delay by systems of ordinary differential equations. This approach follows the foundational work of Krasovskii [**?**, **?**, **?**] regarding the stability and control of systems with hereditary effects.

Section 1. Problem Formulation

Consider the system of differential equations for $t_0 < t \leq T$:

$$\frac{dx(t)}{dt} = \sum_{i=1}^{k} A_i(t)x(t - h_i(t)) \tag{1.1}$$

where $x(t)$ is an n-dimensional vector, $h_i(t)$ are time-varying delays, and $A_i(t)$ are $n \times n$ matrices. The initial condition is given by $x(t) = \phi(t)$ for $t_0 - h \leq t \leq t_0$, where $h = \max_i \sup_t h_i(t)$.

To approximate the delay system (1.1), we introduce a system of ordinary differential equations of dimension $n(m+1)$:

$$\begin{aligned} \frac{dy^{(0)}(t)}{dt} &= \sum_{i=1}^{k} A_i(t) \sum_{j=0}^{m} c_{ij}(t) y^{(j)}(t) \\ \frac{dy^{(1)}(t)}{dt} &= \frac{m}{h}(y^{(0)}(t) - y^{(1)}(t)) \\ &\vdots \\ \frac{dy^{(m)}(t)}{dt} &= \frac{m}{h}(y^{(m-1)}(t) - y^{(m)}(t)) \end{aligned} \tag{1.2}$$

where the coefficients $c_{ij}(t)$ are defined based on the position of the delay $h_i(t)$ within the partitioned interval $[0, h]$. Specifically, $c_{ij}(t) = 1$ if $jh/m \leq h_i(t) < (j+1)h/m$, and $c_{ij}(t) = 0$ otherwise. The initial conditions for (1.2) are set as:

$$y^{(j)}(t_0) = \phi(t_0 - jh/m), \quad j = 0, \dots, m \tag{1.2'}$$

We assume that the delays $h_i(t)$ are continuously differentiable and satisfy the condition:

$$\frac{dh_i(t)}{dt} \leq 1 - \rho, \quad \rho > 0 \tag{1.3}$$

Under these conditions, we aim to show that the solution $y^{(0)}(t)$ of the approximating system (1.2) converges to the solution $x(t)$ of the original system (1.1) as $m \to \infty$.

Section 2. Convergence Analysis

To prove the convergence, we first analyze the case of a single delay:

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)x(t - h(t)) \tag{2.1}$$

The solution to (2.1) can be represented in integral form:

$$x(t) = x(t_0) + \int_{t_0}^{t} A(\tau)x(\tau)d\tau + \int_{t_0}^{t} B(\tau)x(\tau - h(\tau))d\tau \tag{2.2}$$

By performing a change of variables $\tau^* = \tau - h(\tau)$, and utilizing the condition (1.3), we can rewrite the delayed integral term. Let $\tau = f(\tau^*)$ be the inverse function. Then:

$$x(t) = x(t_0) + \int_{t_0}^{t} A(\tau)x(\tau)d\tau + \int_{f(t_0)}^{f(t)} B(f(\tau^*))x(\tau^*)\frac{df}{d\tau^*}d\tau^* \tag{2.3}$$

The approximating system corresponding to (2.1) is:

$$\begin{aligned} \frac{dy^{(0)}(t)}{dt} &= A(t)y^{(0)}(t) + B(t)\sum_{j=0}^{m} c_j(t)y^{(j)}(t) \\ \frac{dy^{(j)}(t)}{dt} &= \frac{m}{h}(y^{(j-1)}(t) - y^{(j)}(t)), \quad j = 1, \dots, m \end{aligned} \tag{2.4}$$

The solution for the intermediate variables $y^{(j)}(t)$ can be expressed using the kernel of the m-th order delay operator:

$$y^{(j)}(t) = \int_{t_0}^{t} \frac{m^j(t-\tau)^{j-1}}{(j-1)!h^j} \exp\left(-\frac{m}{h}(t-\tau)\right) y^{(0)}(\tau)d\tau + \dots \tag{2.5}$$

As $m \to \infty$, the kernel $\frac{m^j(t-\tau)^{j-1}}{(j-1)!h^j}\exp(-\frac{m}{h}(t-\tau))$ behaves like a Dirac delta function centered at $\tau = t - jh/m$. Specifically, using Stirling' s approximation and properties of the Gamma distribution, we observe that for large m:

$$\frac{m^m(t-\tau)^{m-1}}{(m-1)!h^m} \exp\left(-\frac{m}{h}(t-\tau)\right) \to \delta\left(\tau - \left(t - \frac{mh}{m}\right)\right) \tag{2.7}$$

This allows us to approximate the sum $\sum c_j(t)y^{(j)}(t)$ by the delayed state $x(t - h(t))$.

Section 3. Main Results

Theorem 1.1. Let the coefficients $A_i(t)$ be continuous and the delays $h_i(t)$ satisfy condition (1.3). Then for any $\epsilon > 0$, there exists an $M(\epsilon)$ such that for all $m > M$, the solution of the approximating system (1.2) satisfies:

$$\|x(t) - y^{(0)}(t)\| < \epsilon, \quad t \in [t_0, T]$$

Theorem 1.2. If the initial function $\phi(t)$ is Lipschitz continuous, the convergence is uniform on the interval $[t_0, T]$. Furthermore, the intermediate variables $y^{(j)}(t)$ converge to the values of the solution at the corresponding delayed time points:

$$\|y^{(j)}(t) - x(t - jh/m)\| \to 0 \text{ as } m \to \infty \tag{3.1}$$

uniformly for $j = 1, \dots, m$.

The proof relies on the integral representation (2.3) and (2.5). By estimating the difference between the exact delay kernel and the approximating polynomial-exponential kernel, we show that the error term vanishes as m increases. The condition (1.3) ensures that the transformation of the time scale is well-behaved, preventing the "accumulation" of delay effects that could lead to divergence.

References

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Figures

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DIFFERENTIAL EQUATIONS

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ANALYSIS OF A CLASS OF SYSTEMS OF DIFFERENTIAL-DIFFERENCE EQUATIONS BY THE METHOD OF SEPARATION OF MOTIONS

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An approximate method for analyzing systems of equations describing controllers with a digital computer (DC) is presented. The method consists of artificially introducing a "small" parameter into the equations with part of the derivatives (or differences), which allows always lowerly lowering the order of the equations under passsotration and its useɪ to annapate the phase space annapat.

1. Statement of the problem. We will consider the following system of differential equations:

$$\frac{dx}{dt} = Ax + bu_k, \tag{1}$$

where b, x — n-dimensional column vectors; $A = \|a_{ij}\|_n^n$; a_{ij} — are real numbers; u_k — a piecawise-constant control, which formed in the DC and deйctryrating on the time or interval $kT_0 \leqslant t < (k+1)T_0$; T_0 — interpѳall discretization.

Let us assume, that the system (1) is tracking, that is, u_k is calcculated in the DC echode on the meserement of snɪvane $x(t)$ at the moment kT_0, and the calveting time u_k is negligibly mall (combered to T_0). Torga

$$u_k = \phi[x(kT_0)] \quad \text{for} \quad kT_0 \leqslant t < (k+1)T_0, \tag{2}$$

where ϕ — calaпar fynction of the vectora x.

We will solve the following sadlaw: determine the behavior of the traectoriy $x(x_0, t)$ (x_0 — an aroubsolary initial nalune position). Following tradition [1], we reduce the pascmotration of sustems (1) to the uvydy of sustems of differen-ntial paccmenations (1) on the time of interrval T_0 u to the analyze the systems of reccurrent equationing

$$x_{k+1} = Bx_k + du_k \quad (x_k = x(kT_0)),$$

$$B = \exp AT_0), \; d = (\exp AT_0)\int_0^{T_0}\left[\exp - As\right]b\,ds. \tag{3}$$

Custem (3) describes the bobegenor systems (1) at discretive moments kT_0 ($k = 1, 2, 3, \ldots$).

Discyssion of pasrimues metods, for usyvening (1) is contanned in the pa-boɒe [1].

As a prabulo, onpederesing bobegenor custems (1) in enter-sampling un-intervales does ne predestveet trygaent cɛygnoctes. Hɪrforest crownid is the usyvenue custemsi (3).

For the usyding of custems (3) we ucnoльsoase the disckretwid analor metod of separation equɪtation [2].

Figure 1: Figure 1

2. Separation of motions. As in the continuous case, the main moment in the separation of motions is the transformation of the augmented matrix $\|\mathbf{b}_{ij}; \mathbf{d}_i\|$ to a special form.

The following is true

Lemma 1. *Let the system* (3) *be completely controllable* (vectors **d**, **Bd**, ..., $\mathbf{B}^{n-1}\mathbf{d}$ — are linearly independent), then for any $m < n$ *there exists a non-singular linear coordinate trensformation*, reducing (3) to the form

$$z_{i,k+1} = \sum_{j=1}^{n-m} p_{ij}z_{j,k} + p_i z_{n-m+1,k}, \quad i \in \{1, .., n-m\};$$
$$z_{n-m+i,k+1} = \sum_{j=1}^{n-m} q_{ij}z_{j,k} + \sum_{j=1}^{s+1} r_{ij}z_{n-m+j,k}, \quad s \in \{1, .., m-1\}; \tag{4}$$
$$z_{n,k+1} = \sum_{j=1}^{n-m} [q_{nj}z_{j,k} + \sum_{j=1}^{m} r_{nj}z_{n-m+j,k} + d_m u_k,$$

where $d_m \neq 0$, $r_{i,i+1} \neq 0 \quad (i \in \{1, .., m-1\})$.

For the proof of Lemma 1 it is sufficient to consider the auwiliary sustery system

$$\frac{dy}{dt} = By + du$$

and use Lemma 1 of work [2]. In this caye the algorithm of transformation of the augmented υ matrix remains the same as in work [2].

Since it is most covehient to separate two fast motions, then the maining smaining

Lemma 2. *Lct the vectors* **d** *and* **Bd** *be linearly independent, then the system* (3) *by a non-singular linear trensformation is reduced to the form* (4) with $m = 2$.

Proof. From the linear independence of vectors **d** and **Bd** it it follows, that $d_i \neq 0$ and $\sum_{j=1}^{n} b_{ij}d_j \neq 0$ for some $i \in \{1, \dots, n\}$. Let us renumber the coordinates $x_1, x_2, \dots, x_n$, so, that $d_n \neq 0$ and let us apply to (3) the trensformation

$$y = T_1 x, \quad T_1 = \|t_{ij}^{(1)}\|_1^n,$$
$$t_{ij}^{(1)} = \delta_{ij} \quad (i \in \{1, , n\}, j \in \{1, .., n\}), \tag{5}$$
$$t_{in}^{(1)} = -\frac{d_i}{d_n} \quad (i \in \{1, .., n\}) t_{nn}^{(1)} = 1.$$

$(\delta_{ij} = 0$ for $i \neq j, \quad \delta_{ii} = 1)$.

In coordinates y the system (3) takes the firm

$$y_{k+1} = T_1 B T_1^{-1} y_k + d^{(1)} u_k$$
$$(d_i^{(1)} = 0, \quad i \in \{1, ..., n\}, \quad d_n^{(1)} = d_n \neq 0). \tag{6}$$

Let us clarify the structure of the matrixs $T_1 B T_1^{-1}$. For this и represent the matrix in the form of blocks

$$T_1 = \left\| \begin{array}{c|c} E_{n-1} & -d \\ \hline 0_{n-1}^* & 1 \end{array} \right\|; \quad B' = \left\| \begin{array}{c|c} B_{n-1} & b_{in} \\ \hline b_{nj} & b_{nn} \end{array} \right\|; \quad T_1^{-1} = \left\| \begin{array}{c|c} E_{n-1} & d \\ \hline 0_{n-1}^* & 1 \end{array} \right\|,$$

Figure 2: Figure 2

where E_{n-1} — unit matrix of order $(n-1)$; $\mathbf{d}$ — $(n-1)$-dimensional column ($\mathbf{d}_i = -d_i/d_n$, $i \in 1, n-1$); $\mathbf{B}_{n-1}$ — matrix of order n-1; $\mathbf{O}^{\circ}_{n-1}$ — $(n-1)$-dimensional row-vector of zeros; $\mathbf{b}_{jn}$, $\mathbf{b}_{nj}$ — cortersponding colomn-vector and row-vector.

Computing block-wise, we obtain

$$T_1 B T_1^{-1} = \left\| \begin{array}{c|c} B_{n-1} - \mathbf{d}\mathbf{b}_{nj} & \mathbf{q} \\ \hline \mathbf{b}_{nj} & b_{nj}\mathbf{d} + \mathbf{b}_{nn} \end{array} \right\|,$$

where $\mathbf{q} = (B_{n-1} - \mathbf{d}b_{nj})\mathbf{d} + (b_{jn} - b_{nn}\mathbf{d})$.

For the reduction of system (6) to form (4) when $m = 2$, it is necessary and sufficient that $q_i \neq 0$ for at least one $i \in 1, n-1$.

Indeed, in this case, by renembering the first $(n-1)$ coordinates such that $q_{n-1} \neq 0$, and applying transformation T_2 to (6)

$$\mathbf{z} = T_2\mathbf{y}, \quad T_2 = \left\| \begin{array}{c|c} E_{n-2} & -q_{n-2}\mathbf{0}^{\circ}_{n-2} \\ \hline \mathbf{0}^{\circ}_{n-2} & \\ \hline \mathbf{0}^{\circ}_{n-2} & E_2 \end{array} \right\|, \tag{7}$$

$$q_i = q_i/q_{n-1} \quad (i \in 1, n-2), \quad \mathbf{O}^{\circ}_{n-2} = \underbrace{(0, \ldots, 0)}_{(n-2)},$$

we bring (6) to form (4).

Thus, the ycolution for reducing system (3) to form (4) is expressed by the nequenlities

$$d_i \neq 0,$$

$$\sum_{j=1}^{n-1} b_{ij}d_j - \sum_{j=1}^{n-1} d_i b_{nj} d_j + b_{in} - b_{nn}d_i \neq 0 \tag{8}$$

for at least one one $i \in 1, n$—1.

Note, that the stopord of nequenlities (8) for $i = n$ turns into an identity, and scince $d_n \neq 0$, it mame [illegible] than, it man be written in the form $(d_n = 1)$ $\sum_{i=1}^{n} b_{ij}d_j - d_i \sum_{j=1}^{n} b_{nj}d_j \neq 0$ for at least one zone $i \in 1, n$.

The lastlest is equivalent in vector form to the following:

$$\mathbf{Bd} - \mathbf{d}\left(\sum_{j=1}^{n} b_{nj}d_j\right) \neq 0. \tag{9}$$

Inequeality (9) means that linearly nesabecinence of vectors $\mathbf{d}$ and $\mathbf{Bd}$.

Eclu $\mathbf{d}$ and $\mathbf{Bd}$ linearly insabeument, to the left side of (9) is not peal to nyro. Если if $\mathbf{d}$ and $\mathbf{Bd}$ linearly sabeciment, to cymectrsyets taken λ_1 an λ_2, that

$$\lambda_1\mathbf{Bd} + \lambda_2\mathbf{d} = \mathbf{0} \text{ where } \lambda_1^2 + \lambda_2^2 > 0. \tag{10}$$

It follows us (10) that

$$\lambda_1 \sum_{j=1}^{n} b_{ij}d_j + \lambda_2 d_i = 0 \quad (i \in 1, n-1),$$

$$\lambda_1 \sum_{j=1}^{n} b_{nj}d_j + \lambda_2 d_n = 0.$$

Figure 3: Figure 3

But then the corresponding determinants of the 2nd order are equal to zero, i.e.

$$\begin{vmatrix} \sum_{j=1}^{n} b_{ij}d_i & d_i \\ \sum_{j=1}^{n} b_{nj}d_j & d_n \end{vmatrix} = 0 \quad (i \in \overline{1,\ n-1}). \tag{11}$$

Since $d_n \neq 0$, the condition (11) in vector form will take the form

$$Bd = d\left(\sum_{j=1}^{n} b_{nj}\overline{d_j}\right). \tag{12}$$

The contrudeption between (9) and (12) proves the lemma.

More general than lemmas 1, 2, is the following statement: if the vectors **d**, **Bd**, ..., , $\mathbf{B}^{n-1}\mathbf{d}$ are linearly independent, then the system (3) is reduced to the form (4) by a non-singular linear transformation.

The proof of this statement is analogous to the proof of lemma 2, but more cumbersome.

Let us assume that the discretization interval T_0 is sufficiently small. Let us denote $T_0 = \nu$ and we will consider ν as a 'small' parameter. Let us also set $= d_\eta^0 \nu^{-(m-1)}$. As in the case of continuous systems, the latter means that the efficiency of the control force is sufficiently large.

Let us introduce new coordinates $v_1, v_2, \ldots, v_m$ by the formylas

$$v_1 = \nu^{i-1} z_{n-m+1} \quad (i \in \overline{1,\ m}). \tag{13}$$

Teren, consiteiring that $d_n = d_n^0 \nu^{-(m-1)}$, we holyin

$$\begin{gathered} z_{k+1} = Pz_k + pv_{1k} \quad (z^* = (z_1, z_2, \ldots, z_{n-m})), \\ v_{k+1} = Q_1 z_k + R_0 v_k + d_n^0 e^{(m)} u_k, \quad P = \|p_{ij}|_{n-m}^{n-m}, \end{gathered} \tag{14}$$

rde

$$e_i^{(m)} = 0 \quad (i \in \overline{1,\ m-1}), \quad e_i^{(m)} = 1, \quad R_0 = T_y R T_\nu^{-1};$$

$$Q_1 = T_\nu Q,$$

$$T_\nu = \left\| \underbrace{\begin{matrix} 1 & 0 & \ldots & 0 \\ 0 & \nu & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \nu^{m-1} \end{matrix}}_{m} \right\|,$$

$$R_0 = \left\| \begin{matrix} r_{11} & r_{11}\nu^{-1} & 0 & \ldots & 0 \\ \nu r_{21} & r_{22} & \nu^{-1} r_{23} & \ldots & 0 \\ \nu^{m-1} r_{m-1} & \ldots & \ldots & \ldots & r_{mm} \end{matrix} \right\|.$$

It is not difficult to see that as $\nu \to 0$, $r_{ii} + 1 = O(\nu)$, $i \in 1, n-1$. Indeed,

$$B = \exp AT_0 = E + A\nu + O(\nu^2)$$

as $\nu \to 0$ and, therefore, all elements of the matriux B, except those on the main diagonal, are of order of neriux ν. As shown above,

$$q_i = \sum_{j=1}^{n-1} b_{ij}d_j - d_i \sum_{j=1}^{n} b_{aj}d_j =$$

Figure 4: Figure 4

$$= b_{ii}\bar{d}_i - b_{nn}\bar{d}_i + O(\nu) = O(\nu) \text{ as } \nu \to 0.$$

This means that the element $r_{m-1,m}$ has an order of smallness with respect to ν equal to 1. Let us note that as $\nu \to 0$, system (4) transforms into the corresponding system of differential equations, which will have at least m controllable coordinates. It follows that the order of smallness of $r_{i,\,i+1}$ $(i = 1,\, m-1)$ is exactly equal to 1 and

$$r^0_{i,\,i+1} = r_{ii+1}\nu^{-1} \neq 0.$$

Therefore, system (14) can be written in the form

$$z_{k+1} = z_k + \nu P_0 z_k + \nu p_0 \nu_{1,k},$$
$$\nu_{k+1} = Q_1 z_k + R_0 \nu_k + d^0_n e^{(m)} u_k, \tag{15}$$

where $r^0_{i,\,i+1} \neq 0$ and $p_0 = O(1)$, $r^0_{ii+1} = O(1)$ as $\nu \to 0$.

From the form of system (15), it follows that for small ν, the increment z_k will have an order of smallness ν, while the increment ν_k will remain comparable to 1. The latter gives the right to call the coordinates $\nu_1, \nu_2, \ldots, \nu_m$ fast, and the coordinates $z_1, z_2, \ldots, z_{n-m}$ slow.

Thus, in the case of the presence of m controllable coordinates and a sufficiently large value of b in system (3), it is possible to separate m 'fast' motions from $(n-m)$ 'slow' ones.

3. Asymptotic representation of the solution. For the investigation (or calculation) of the solution of system (15), it is convenient to use the following method of successive approximations:

$$o)\quad \begin{aligned} z^{(0)}_{k+1} &= z_0 = \text{const},\\ \nu^{(0)}_{k+1} &= Q_1 z_0 + R_0 \nu^{(0)}_k + d^0_n e^{(m)} u_k\,(z^{(0)}_k,\ \nu^{(0)}_k);\end{aligned}$$
$$i)\quad \begin{aligned} z^{(i)}_{k+1} &= P z^{(i)}_k + \nu\, p_0 \nu^{(i-1)}_{1k}\big|_{z^{(i-1)}_k \to z^{(i)}_k},\\ \nu^{(i)}_{k+1} &= Q_1 z^{(i)}_k + R_0 \nu^{(i)}_k + d^0_n e^{(m)} u_k\,(z^{(i)}_k,\ \nu^{(i)}_k).\end{aligned} \tag{16}$$

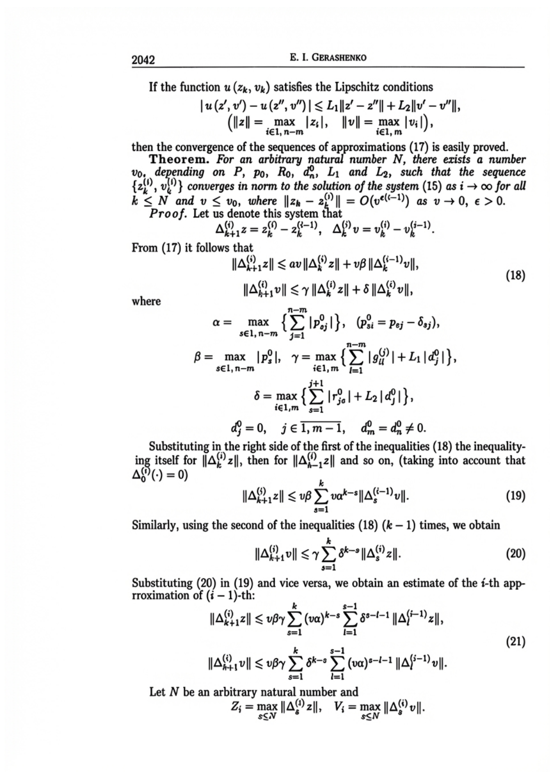
Here, the symbol $\nu^{(i-1)}_{1k}\big|_{z^{(i-1)}_k \to z^{(i)}_k}$ means the following: the second of the systems for determining the $(i-1)$-th approximation will determine the value $\nu^{(i-1)}_{1,k}$, which depends explicitly on $z^{(i-1)}_k$. To obtain the i-th approximation, it is necessary to substitute $\nu^{(i-1)}_{1k}$, having previously replaced $z^{(i-1)}_k$ with $z^{(i-1)}_k$.

The connessive mochannous. The successive approximations (16) give an asymptotic (as $\nu \to 0$) representation of the solution of system (15) (or (4)).

For practical purposes, it is sufficient to limit oneself to the first approximation. A weaker convergence than (16), but a simpler calculation algorithm is provided by the usual method of successive approximations:

$$o)\quad \begin{aligned} z^{(0)}_{k+1} &= z_0 = \text{const},\\ \nu^{(0)}_{k+1} &= Q_1 z_0 + R_0 \nu^{(0)}_k + d^0_n e^{(m)} u\,(z^{(0)}_k,\ \nu^{(0)}_k),\end{aligned}$$
$$i)\quad \begin{aligned} z^{(i)}_{k+1} &= P z^{(i)}_k + \nu\, p_0 \nu^{(i-1)}_{1k},\\ \nu^{(i)}_{k+1} &= Q_1 z^{(i)}_k + R_0 \nu^{(i)}_k + d^0_n e^{(m)} u_k.\end{aligned} \tag{17}$$

Figure 5: Figure 5



If the function $u(z_k, v_k)$ satisfies the Lipschitz conditions

$$|u(z', v') - u(z'', v'')| \leqslant L_1\|z' - z''\| + L_2\|v' - v''\|,$$
$$\Big(\|z\| = \max_{i\in 1,\, n-m} |z_i|, \quad \|v\| = \max_{i\in 1,\, m} |v_i|\Big),$$

then the convergence of the sequences of approximations (17) is easily proved.

Theorem. *For an arbitrary natural number N, there exists a number v_0, depending on P, p_0, R_0, d_n^0, L_1 and L_2, such that the sequence $\{z_k^{(i)}, v_k^{(i)}\}$ converges in norm to the solution of the system (15) as $i \to \infty$ for all $k \le N$ and $v \le v_0$, where $\|z_k - z_k^{(i)}\| = O(v^{\epsilon(i-1)})$ as $v \to 0$, $\epsilon > 0$.*

Proof. Let us denote this system that

$$\Delta_{k+1}^{(i)} z = z_k^{(i)} - z_k^{(i-1)}, \quad \Delta_k^{(i)} v = v_k^{(i)} - v_k^{(i-1)}.$$

From (17) it follows that

$$\|\Delta_{k+1}^{(i)} z\| \leqslant av\,\|\Delta_k^{(i)} z\| + v\beta\,\|\Delta_k^{(i-1)} v\|,$$
$$\|\Delta_{k+1}^{(i)} v\| \leqslant \gamma\,\|\Delta_k^{(i)} z\| + \delta\,\|\Delta_k^{(i)} v\|, \tag{18}$$

where

$$\alpha = \max_{s\in 1,\, n-m} \Big\{\sum_{j=1}^{n-m} |p_{sj}^0|\Big\}, \quad (p_{si}^0 = p_{sj} - \delta_{sj}),$$

$$\beta = \max_{s\in 1,\, n-m} |p_s^0|, \quad \gamma = \max_{i\in 1,\, m} \Big\{\sum_{l=1}^{n-m} |g_{il}^{(j)}| + L_1\,|d_j^0|\Big\},$$

$$\delta = \max_{i\in 1, m} \Big\{\sum_{s=1}^{j+1} |r_{js}^0| + L_2\,|d_j^0|\Big\},$$

$$d_j^0 = 0, \quad j \in \overline{1, m-1}, \quad d_m^0 = d_n^0 \neq 0.$$

Substituting in the right side of the first of the inequalities (18) the inequality-ing itself for $\|\Delta_k^{(i)} z\|$, then for $\|\Delta_{k-1}^{(i)} z\|$ and so on, (taking into account that $\Delta_0^{(i)}(\cdot) = 0$)

$$\|\Delta_{k+1}^{(i)} z\| \leqslant v\beta \sum_{s=1}^{k} v\alpha^{k-s}\|\Delta_s^{(i-1)} v\|. \tag{19}$$

Similarly, using the second of the inequalities (18) $(k-1)$ times, we obtain

$$\|\Delta_{k+1}^{(i)} v\| \leqslant \gamma \sum_{s=1}^{k} \delta^{k-s}\|\Delta_s^{(i)} z\|. \tag{20}$$

Substituting (20) in (19) and vice versa, we obtain an estimate of the i-th app-rroximation of $(i-1)$-th:

$$\|\Delta_{k+1}^{(i)} z\| \leqslant v\beta\gamma \sum_{s=1}^{k} (v\alpha)^{k-s} \sum_{l=1}^{s-1} \delta^{s-l-1}\,\|\Delta_l^{(i-1)} z\|,$$
$$\|\Delta_{k+1}^{(i)} v\| \leqslant v\beta\gamma \sum_{s=1}^{k} \delta^{k-s} \sum_{l=1}^{s-1} (v\alpha)^{s-l-1}\,\|\Delta_l^{(i-1)} v\|. \tag{21}$$

Let N be an arbitrary natural number and

$$Z_i = \max_{s\le N} \|\Delta_s^{(i)} z\|, \quad V_i = \max_{s\le N} \|\Delta_s^{(i)} v\|.$$

Figure 6: Figure 6

Then from (21) it follows

$$Z_i \leqslant vfZ_{i-1} \quad \left(f = \frac{\beta\gamma}{\delta - 1}\left[\delta\,\frac{\delta^N - (v\alpha)^N}{\delta - v\alpha} - \frac{1-(v\alpha)^N}{1 - v\alpha}\right]\right),$$
$$V_i \leqslant v\varphi V_{i-1} \quad \left(\varphi = \frac{\beta\gamma}{v\alpha - 1}\left[\frac{\delta^N - (v\alpha)^N}{\delta - v\alpha} - \frac{\delta^N - 1}{\delta - 1}\right]\right). \tag{22}$$

Obviously, that for vf, $v\varphi$, smaller than 1 (by the way, f and φ are always >0), the sequences $\{Z_i\}$, $\{V_i\}$ monotonically converge to zero. Moreover for fixed N the values f and φ are finite, and if $f \leqslant f_0 v^{-(1-\epsilon)}$, then $Z_i = O(v^\epsilon)Z_{i-1}$. Similarly if for sufficiently small v $\varphi \leqslant \varphi_0 v^{-(1-\epsilon)}$, then

$$V_i \leqslant \varphi_0 v^\epsilon V_{i-1} \leqslant \ldots \leqslant v^{i\epsilon}\varphi_0^i V_0.$$

Theorem proved.

Note, that for convergence for any N it is sufficient, that the value δ was smaller than 1. (In this case $\epsilon = 1$).

The proved theorem establishes the fact, that successive approximations (17) asymptotically converge to the solution of system (14) (or (4)). The obtained conditions for convergence are quite rigid, and the main thing are inapplicable for the case of discontinuous control. More promising, in our view, is the solution of the problem on stability of the solution of system (15) with respect to the first approximation, defined by formulas (16) or (17).

4. Example. As an illustration let us consider the following system:

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = x_3, \quad \frac{dx_3}{dt} = -Ku_k(x). \tag{23}$$

Obviously,

$$B = \exp AT_0 = \begin{vmatrix} 1 & T_0 & \frac{1}{2}T_0^2 \\ 0 & 1 & T_0 \\ 0 & 0 & 1 \end{vmatrix}; \qquad d = -K\begin{vmatrix} \frac{T_0^3}{3!} \\ \frac{T_0^2}{2!} \\ T_0 \end{vmatrix}$$

and system (3) will take the form

$$x_{k+1} = Bx_k + du_k. \tag{24}$$

Transformation T_1 consists of the following:

$$y_1 = x_1 - \frac{T_0^2}{6}x_3, \quad y_2 = x_2 - \frac{T_0}{2}x_3, \quad y_3 = x_3, \tag{25}$$

and in coordinates y system (24) will be written as follows:

$$y_{k+1} = \begin{vmatrix} 1 & T_0 & T_0^2 \\ 0 & 1 & T_0 \\ 0 & 0 & 1 \end{vmatrix} y_k - K\begin{vmatrix} 0 \\ 0 \\ T_0 \end{vmatrix} u_k. \tag{26}$$

Transformation $T_2\,(z = T_2 y)$ will be expressed by the formulas:

$$z_1 = y_1 - T_0 y_2, \quad z_2 = y_2, \quad z_3 = y_3, \tag{27}$$

Figure 7: Figure 7

and in z coordinates the system will take the form

$$z_{k+1} = \begin{vmatrix} 1 & T_0 & 0 \\ 0 & 1 & T_0 \\ 0 & 0 & 1 \end{vmatrix} z_k - K \begin{vmatrix} 0 \\ 0 \\ T_0 \end{vmatrix} u_k. \tag{28}$$

Let $m = 2$, $K = \nu^{-2}K_0$, $T_0 = \nu$, $v_1 = z_2$, $v_2 = \nu z_3$. Then system (14) will convert to the system

$$\begin{aligned} z_{1,k+1} &= z_{1,k} + v_{11,k}, \\ v_{1,k+1} &= v_{1,k} + v_{2,k}, \\ v_{2,k+1} &= v_{2,k} - \bar{K}_0 u_k. \end{aligned} \tag{29}$$

Note that $z = T_2 T_1 \boldsymbol{x}$ and

$$T_2 T_1 = \begin{bmatrix} 1 & -T_0 & \frac{T_3^2}{3} \\ 0 & 1 & -\frac{T_0}{2} \\ 0 & 0 & 1 \end{bmatrix}; \quad T_1^{-1} T_2^{-1} = \begin{bmatrix} 1 & T_0 & \frac{1}{6} T_0^2 \\ 0 & 1 & \frac{1}{2} T_0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let us investigate the case of relay control

$$u_k = \operatorname{sign} x_{1k}.$$

Then in coordinates z, v

$$u_k = \operatorname{sign} \sigma_k, \quad \sigma_k = z_{1k} + \nu v_{1,k} + \frac{\nu}{6} v_{2,k}.$$

The equations of the zero approximation will take the form

$$\begin{aligned} z_k^{(0)} = z_{10}, \quad v_{2,k+1}^{(0)} &= v_{1,k}^{(0)} + v_{2,k}^{(0)}, \\ v_{2,k+1}^{(0)} &= v_{2,k}^{(0)} - K_0 \operatorname{sign} \sigma_k^{(0)}. \end{aligned} \tag{30}$$

The trajectories of system (30) lie in the place $z_1 = z_{10}$ and pass though the toints of parabolas

$$\begin{aligned} 2K_0 v_1 + \left(v_2 + \frac{K_0}{2}\right)^2 &= \text{const, at } \sigma > 0, \\ -2K_0 v_1 + \left(v_2 - \frac{K_0}{2}\right)^2 &= \text{const, at } \sigma < 0. \end{aligned} \tag{31}$$

Due to time quantization, switching can occur at many point of the sectors S_1 and S_2, enclosed respectively between the lines:

$$S_1: \quad \sigma^{(0)} = 0; \quad z_1 + \nu v_1 - \frac{5}{6} \nu v_2 = -\frac{5}{6}, K_0; v_2 > 0,$$

$$S_2: \quad \sigma^{(0)} = 0; \quad z_{10} + \nu v_1 - \frac{5}{6} \nu v_2 = \frac{5}{6}, K_0; v_2 < 0 \tag{32}$$

$$\left(\sigma^{(0)} = z_1 + \nu v_1 + \frac{\nu}{6} v_2\right).$$

Figure 8: Figure 8

Considering the point transformations S_1 to S_2 and S_2 to S_1 by virtue of the system (30), it can be shown that any point of the sector S_1 with coordinate v_{20} in one revolution relative to the point $v_2 = 0$, $v_1 = -z_{10}/v$ will have the coordinate $v_2'' > v_{20}$. Consequently, from a qualitative standpoint, the zero approximation of fast motions (motions in the place (v_1, v_2)) will represent an unwinding spiral relative to the point $v_2 = 0$, $v_1 = -\frac{z_{10}}{v}$.

Since "on average" (over the time of one revolution in the plane (v_1, v_2)) the sign of v_1 is opposite to the sign of z_{10}, the first approximation to z_1 will decrease in absolute value, but at the moment of vanishing $z_{1,k}$ the coordinates v_1 and v_2 will have large values. This will lead to the fact that the sign of the coordinate z_1 will change to the opposite and the magnitude $|z_{1,k}|$ will increase approximately until the next switching. Due to the fact that the fast motions represent an unwinding spiral, the transient process for z_1 represents divergent oscillations with increasing length of the half-wave.

Thus, even the consideration of the zero approximation makes it possible to conclude on the qualitative picture of motion in the system (23) (or (29)): the zero solution of the system is unstable.

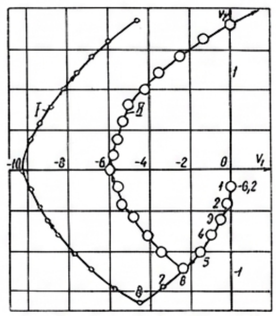


Fig. 1. Phase portrait of fast motions: I — curve of zero approximation; II — curve of first approximation, also the experimental (calculated on a digital computer) trajectory of fast motions

In Fig. 1, a portrait of fast motions in the plane (v_1, v_2) is shown, obtained by the method of successive approximations manually and calculated on a digital computer, and in Fig. 2, a graph of changes in $x_{1,k}$, obtained on a digital computer, is shown. The calculation data fully confirm conclusion about the qualitative character of the motion.

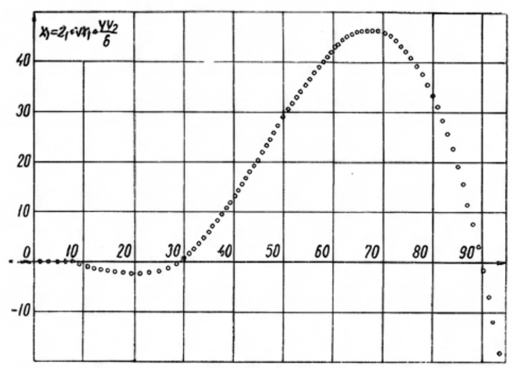


Fig. 2. Results of the calculation on a digital computer of the transient process for x_1

Figure 9: Figure 9

Conclusions

1. The method of separation of motions for discrete control systems allows ectablishing the qualitative character of motion under a nonlinear form of control action.
2. The form of control $u_k(x)$ does not play such a significant role, ras in in known methods [1], due to the reduction of order and the possibility of using the phase plane apparatus.
3. Under the method of separation of motions, a nonlinear system is regarded as essentially nonlinear.

The author thanks profescor E. A. Barbashin for his attention to this work.

Note during proofreading. As suclequent investigations and palculation of conkrete examples, carried out by the author and V. M. Reshetov, have shown, the method set forth in the article yields satisfactorary connacty for a suctatiovily large sampling interval — T_0 (more precisely, when when $d_n^0 = d_n/\nu^{(m-1)} = O(1)$). If d_n^0 is mall (for example, $d_n^0 = O(\nu)$, then it is expedient first to separate the motions in the system of differential equations (1), and then to proceed to the recurrent system of type (3).

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Figure 10: Figure 10

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The further course of reasoning is the same as in [1]. We present it briefly for completeness of exposition. From Lemma 2.1 it follows that the quantities

$$Q_m(t,\tau)=\begin{cases}\int\limits_{\tau}^{t} B(\nu)\chi_{[t_1,m]}(\nu,\tau)\,d\nu, & \text{if } c_0^{(m)}(\tau)=0,\\ B(\tau), & \text{if } c_0^{(m)}(\tau)=1,\end{cases}$$

are uniformly orauned with respect to m and converge in measure [4] to the $\times(1-h'(\tau^*(\tau)))^{-1}\chi(t,\tau)$. From Lemma 2.2 it follows in turn that the em pahnomerno no m organiovencise quantities

$$Q_m^*(t,\nu)=\int\limits_{t_0}^{t} B(\tau)\chi^*_{[t_1,m]}(\nu,\tau)\,d\tau$$

converge in measure to the function $B((\nu))(1-h'(\nu^*(\nu)))^{-1}\chi^*(t,t_0,\nu)$. In boun chyvares exodimoctis maise pahnomepra no t.

Corollary **2.1.** The solutions $x(t)$ of system 2.1 *are uniformly brauned with respect* to acll initial functions with norms $\|\varphi(\nu)\|_{[-h,0]}\leqslant 1$. *Fsolutions* $y^{(0)}(t)$ of system (2.4) *are uniformly brauned with respect to* all numbers m and all initial vectors $y^{(i)}(t_0)$, $i=0,\ldots,m$, correrstonding by virtue of equelties (1.3) to initialise functions $\varphi(\nu)$ with normins $\|\varphi(\nu)\|_{[-h,0]}\leqslant 1$.

The indicated properties follow from the uniform organiness of the fynction $\chi[t,\tau]$, $\chi^*[t,t_0,\nu]$, $Q_m(t,\tau)$, $Q_m^*(t,\nu)$ and are verified directly from the intergrapheical equationens (2.3), (2.5).

Setening $z(t)=y^{(0)}(t)-x(t)$, we bolyain from expressions (2.3), (2.5) an integral equation of Volterra type [5] for the functions $z(t)$

$$H(t)z(t)=R_m(t), \qquad (2.27)$$

where $H(t)$ denates the linear operator

$$H(t)=z(t)-\int\limits_{t_0}^{t}A(\tau)z(\tau)\,d\tau-\int\limits_{t_0}^{t}Q_m(t,\tau)z(\tau)\,d\tau,$$

$$R_m(t)=\int\limits_{t_0}^{t}\{Q_m(t,\tau)-\chi[t,\tau]B(\tau^*)(1-h'(\tau^*))^{-1}\}\,x(\tau)\,d\tau+$$

$$+\int\limits_{t_0-h(t_0)}^{t}\{Q_m^*(t,\nu)-\chi[t,t_0,\nu]\,B(\nu^*)(1-h'(\nu^*))^{-1}\}\,\varphi(\nu)\,d\nu.$$

Fynktion $R^{(m)}(t)$ in this cayae possesses the oronstry, that $\|R^{(m)}(t)\|^{(2)}_{L_{[t_0,T]}}\to 0$ as $m\to\infty$ uniformly respect to acel functions $\varphi(\nu)$ with norm $\|\varphi(\nu)\|_{[-h,0]}\leqslant 1$. The latter follows from Lemmas 2.1, 2.2 and Corollary 2.1. The operator $H(t)$ hraes an inverse operator $H^{-1}(t)$, organuvously uniformly to m and $t_0\leqslant\leqslant t\leqslant T$. Considering tnn the quantities $z(t)$, $R_m(t)$ as elements of the peoctess $L_2[0,T]$, we find, that $\|z(t)\|^{(2)}_{L_{[t_0,T]}}\leqslant\|H^{-1}\|^{(2)}_{L_{[t_0,T]}}\|R_m(t)\|^{(2)}_{L_{[t_0,T]}}$. otkyda it follow egasy, that $\|z(t)\|^{(2)}_{[t_0,T]}\to 0$ uniforming to t, t_0 and with functions $\varphi(\nu)$ with norm $\|\varphi(\nu)\|_{[-h,0]}\leqslant 1$. The latter property ensures also also the following relation:

$$\lim_{m\to\infty}\|z(t)\|^{C}_{[t_0,T]}\to 0,$$

Figure 11: Figure 11

uniform in $t_0 \leqslant T$, $\| \varphi(\theta) \|_{[-h, 0]} \leqslant 1$. It is verified by equation (2.27) taking into account the Cauchy–Bunyakovsky inequality, where the proof of Theorem 2.1 for equation (2.1) ends.

The proof of Theorem 2.1 for system (1.1) is carried out in the same way as for system (2.1). This time, however, in equations (2.3), (2.5) there will appear several additional integral terms, corresponding to the number of available delays. The mentioned terms are estimated completely uniformly, as a result of which we finally obtain again an equation of the type (2.27), from which Theorem 2.1 follows.

§ 3. PROOF OF THEOREM 1.2

Let us consider again equation (2.1) and its approximating system (2.4) with initial data (1.3). It follows from Theorem 2.1 that $\| x(t) - y^{(0)}(t) \|_{[t_0, T]} \to \infty$ as $m \to \infty$, $\| \varphi(\theta) \|_{[-h, 0]} \leqslant 1$. It follows from this that $\| y^{(0)}(t-\xi) \| \to \| x(t-\xi) \|$ as $m \to \infty$, whatever t, $0 \leqslant \xi \leqslant 1$ are. But then for the proof of Theorem 1.2 it is sufficient to check the following assertion.

Lemma 3.1. *If $h'(t) < 1 - \rho$, then for any $\varepsilon > 0$, $\delta > 0$ one can indicate such an $M(\varepsilon, \delta, \rho) < \infty$, that for all $m > M(\varepsilon, \delta, \rho)$ the following inequality will be fulfilled*

$$\| y^{(i)}(t) - y^{(0)}(t - i/m) \| \leqslant \varepsilon, \tag{3.1}$$

uniform in all $t_0 \leqslant t \leqslant T$ and all $i = 1, \ldots, i'$; $i' = [mh(t)/h]$. Let's provere the lemma. For this purpose, we note that integration of the system (2.4) gives the condition

$$y^{(i)}(t) = \frac{m^t}{(i-1)!} \int_{t_0}^{t} y^{(0)}(\tau)(t-\tau)^{i-1} \exp(-m(t-\tau))\, d\tau +$$

$$+ \sum_{k=1}^{m} y^{(k)}(0) \frac{[m(t-i_0)]^{i-k}}{(i-k)!} \exp(-m(t-t_0)), \tag{3.2}$$

Let's transform the last term taking into account Stirling's formula and equalities (1.2′). Then we obtain

$$\sum_{k=1}^{i} y^{(k)} \frac{[m(t-t_0)]^{i-k}}{(i-k)!} \exp(-m(t-t_0)) = \int_{-1}^{0} \varphi(\theta) \psi_m^{(i)}(\theta, t)\, d0, \tag{3.3}$$

where

$$\psi_m^{(i)}(\theta, t) = \left[\frac{m(t-t_0)}{i-k} \exp\left(\frac{-m(t-t_0)}{i-k} + 1 \right) \right]^{i-k} \times$$

$$\times (2\pi(i-k))^{-1/2} \exp(-\theta_{i-k}) \quad \text{при} \quad \theta \leqslant \left[-\frac{k}{m}, -\frac{k-1}{m} \right),$$

$$k = 1, \ldots, i; \qquad \psi_m^{(i)}(\theta, t) = 0 \quad \text{при} \quad -1 \leqslant 0 < -i/m.$$

Having chosen t such that $t > t_0 + 1 + \delta$, we see that $t > \vartheta^*(\theta)$, whatever θ is from $-1 \leqslant \theta < 0$. Involving further the reasoning used in the derivation of (2.20′), (2.21), it can be verified that as $m \to \infty$ for any $\omega' > 0$ the condition condition $m \psi_m^{(i)}(\theta, t) \to 0$ will be fulfilled uniformly in all $i = 1, \ldots, m$; $t > t_0 + 1 + \delta$ and all θ from $[-1, 0]$, with the exception only of a set of measure

Figure 12: Figure 12

less than w'. Consitoring the condition $\|\varphi(\theta^{\circ})\|_{[-h,0]} \leqslant 1$, we finally obtain train, that for only $\varepsilon > 0$ one mon choose such a large $n_1(s)$, thot, such for $m > n_1(s)$

$$\Big\| \int_{-1}^{0} \varphi(\theta)\, m\, \psi^{(i)}(\theta, t)\, d\theta \Big\| < \varepsilon. \tag{3.4}$$

Let $\delta \leqslant i/m \leqslant h(t)$. Let us transform the integral

$$\frac{m^i}{(i-1)!} \int_{t_0}^{t} y^{(0)}(\tau)(t-\tau)^{i-1} \exp(-m(t-\tau))\, d\tau,$$

uomasing, such as in (2.3), formling's Stirling. Splitting the intergal into parts

$$\int_{t_1-\sigma}^{t} + \int_{t_1-\sigma}^{t_1+\sigma} + \int_{t_0}^{t_1-\sigma}, \qquad \sigma < 58, \qquad t_1 = t - i/m,$$

we nolyw further that as $m \to \infty$ intergals $\int_{t_1-\sigma}^{t}$, $\int_{t_0}^{t_1-2}$ tend to zero uahio prsomeiply for scex $t_0 \leqslant t \leqslant T$ and, moreover,

$$\int_{t_1-\sigma}^{t_1+\sigma} \to y(t - i/m) + o(m, \sigma),$$

where $o(m, \sigma) \to 0$ as $m \to \infty$, $\sigma \to 0$. The located propertietch with the help of estimates similar to those used in the proof of Lemma 2.1. Thus, taking into account (3.4), we conclude, that Lemma 3.1 is valid verified at least for $i/m > \delta$. Further, we note, that, tho for any $\varepsilon > 0$ one can choose $\delta > 0$ such that сuch.a that $\|y^{(0)}(t) - y^{(0)}(t-\xi)\| \leqslant \varepsilon$ as long as o $0 < \xi < 8$. The locteder follows us the unifiorm in m organuveness of fynkuion $y^{(0)}(t)$ on the integvale $0 \leqslant t \leqslant T$. Ohonatily, we opperemine the nuclon m, appearing in Lemma 2.2, buiбoping for $\varepsilon > 0$ nuclon $\delta > 0$ and sen by snate- and then by choosing δ — number m.

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Figure 13: Figure 13