ON P. P. KOROVKIN' S THEOREMS

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Abstract

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Bl. SENDOV

ON P. P. KOROVKIN' S THEOREMS FOR THE CONVERGENCE OF SEQUENCES OF LINEAR POSITIVE OPERATORS

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The purpose of the present communication is to show how some theorems of P. P. Korovkin ([1]) can be generalized, and to prove analogous assertions on the convergence of sequences of linear positive operators with respect to the Hausdorff distance ([2]).

1. Let $f(x)$ be a function defined on an interval Δ (finite or infinite). We shall call the completed graph $\bar{f}$ of the function $f(x)$ the intersection of all closed and convex, with respect to the y-axis, point sets in the plane that contain the graph of the function $f(x)$. The completed graph of every function is a closed and connected point set in the plane. The completed graph of a continuous function coincides with its graph.

2. Let $f(x)$ and $g(x)$ be bounded functions defined on one and the same interval. The Hausdorff distance between $f(x)$ and $g(x)$ will be called

$$r(f,g) = \max\left\{\max_{X\in\bar{f}}\min_{Y\in\bar{g}}\|X-Y\|_0,\ \max_{X\in\bar{g}}\min_{Y\in\bar{f}}\|X-Y\|_0\right\},$$

where

$$\|X-Y\|_0 = \|X(x_1,y_1)-Y(x_2,y_2)\|_0 = \max\left[|x_1-x_2|,\ |y_1-y_2|\right].$$

The following theorem holds ([3]):

Theorem 1. *If a sequence of functions $\{f_n(x)\}$, defined on an interval Δ, converges with respect to the Hausdorff distance to a continuous function $f(x)$, then the sequence $\{f_n(x)\}$ converges to $f(x)$ uniformly on the interval Δ.*

3. By the modulus of nonmonotonicity $\mu_f(\delta)$ of the function $f(x)$ we shall mean

$$\mu_f(\delta) = \sup_{|x_1-x_2|\le\delta} \left\{ \sup_{x_1\le x\le x_2} [|f(x_1)-f(x)|+|f(x_2)-f(x)|] - |f(x_1)-f(x_2)| \right\}.$$

We shall call the function $f(x)$ locally monotone if its modulus of nonmonotonicity satisfies the condition

$$\lim_{\delta\to 0} \mu_f(\delta) = \mu_f(0) = 0.$$

It is not difficult to show that the necessary and sufficient condition for $f(x)$ to be locally monotone is the following: $f(x)$ may have only discontinuities of the first kind, and for every x the value $f(x)$ lies between $f(x-0)$ and $f(x+0)$. Every continuous function is locally monotone.

4. We shall formulate two theorems which, by virtue of Theorem 1, are generalizations of the corresponding theorems of P. P. Korovkin ([1]).

Theorem 2. *Let the following conditions be satisfied for a sequence of linear positive operators* $\{L_n(f;x)\}$*:*

$$L_n(1;x) = 1 + \alpha_n(x),$$

$$L_n(t;x) = x + \beta_n(x),$$

$$L_n(t^2;x) = x^2 + \gamma_n(x),$$

where $\alpha_n(x), \beta_n(x), \gamma_n(x)$ tend uniformly to zero on the finite interval $\Delta = [a,b]$. If the function $f(x)$ is defined on the interval Δ, is locally monotone on this interval, is continuous at the points a and b, and belongs to the domain of definition of the operators under consideration, then the sequence $\{L_n(f;x)\}$ converges to the function $f(x)$ with respect to the Hausdorff distance.

Theorem 3. Let the following conditions be satisfied for the sequence of linear positive operators $\{L_n(f;x)\}$:

$$L_n(1;x) = 1 + \alpha_n(x),$$
$$L_n(\cos t;x) = \cos x + \beta_n(x),$$
$$L_n(\sin t;x) = \sin x + \gamma_n(x),$$

where $\alpha_n(x), \beta_n(x), \gamma_n(x)$ tend uniformly to zero for all x. If the function $f(x)$ is locally monotone, 2π-periodic, and belongs to the domain of definition of the operators under consideration, then the sequence $\{L_n(f;x)\}$ converges to the function $f(x)$ with respect to the Hausdorff distance.

Concerning an estimate of the rate of convergence of sequences of linear positive operators, one can state the following assertion.

Theorem 4. Let the following conditions be satisfied for the sequence of linear positive operators $\{L_n(f;x)\}$:

$$L_n(1;x) = 1,$$
$$L_n(\cos t;x) = \cos x + O(\lambda_n^3),$$
$$L_n(\sin t;x) = \sin x + O(\lambda_n^3),$$

where $\{\lambda_n\}$ is a sequence of nonnegative numbers for which

$$\lim_{n\to\infty} \lambda_n = 0.$$

If the function $f(x)$ is 2π-periodic, locally monotone with modulus of nonmonotonicity $\mu_f(\delta)$, and belongs to the domain of definition of the operators under consideration, then

$$r(L_n(f;x), f(x)) \leq \tfrac{1}{2}\mu_f(\lambda_n) + C\lambda_n,$$

where C is a constant independent of n.

There is also an analogue of this theorem for uniform convergence, where λ_n^2 appears instead of λ_n^3. But with the method of proof used by us, a better estimate cannot be obtained.

We give one more theorem, which better estimates the order of approximation and may be used in some cases to find estimates, exact in order, for concrete operators.

Theorem 5. Let $L(f;x)$ be a linear positive operator for which

$$L(1;x) = 1,$$
$$r\big(L(\sigma(t+\alpha);x), \sigma(x+\alpha)\big) \leq \lambda < 1,$$

where $\sigma(t) = \operatorname{sgn}\sin t$, α is an arbitrary number, and λ is a nonnegative constant. If $f(x)$ is a 2π-periodic function, locally monotone with modulus of nonmonotonicity $\mu_f(\delta)$, and belonging to the domain of definition of the operator $L(f;x)$, then

$$r(L(f;x), f(x)) \leq \frac{1+\lambda}{2}\,\mu_f(4\lambda+0) + \frac{\lambda}{1+\lambda}\max[1, 2M],$$

or, for $\lambda \leq \frac{1}{2}$ and $M = \sup_x |f(x)| \geq \frac{1}{2}$,

$$r(L(f;x), f(x)) \leq \mu_f(4\lambda + 0) + 4\lambda M.$$

We note that from the local monotonicity and periodicity of $f(x)$ it follows that it is bounded, i.e. $M < \infty$.

If one considers integral operators with a positive and symmetric kernel

$$k(f;x) = \int_{-\pi}^{\pi} f(x+t)K(t)\,dt; \qquad K(t) \geqslant 0, \qquad K(-t) = K(t),$$

then for them

$$k(f(t+\alpha);x) = k(f(t);x+\alpha),$$

and λ in Theorem 5 is computed by the formula

$$\lambda = \inf_{\delta>0} \max\left[\delta,\ 2\int_{\delta}^{\pi} K(t)\,dt + 2\int_{\pi-\delta}^{\pi} K(t)\,dt\right].$$

Thus, for example, for Jackson' s operator $U_n(f;x)$ the corresponding λ is equal to

$$\lambda_n = \inf_{\delta>0} \max\left[\delta,\ \frac{3}{\pi n(2n^2+1)}\left(\int_{\delta}^{\pi} + \int_{\pi-\delta}^{\pi}\right)\left(\frac{\sin nt/2}{\sin t/2}\right)^4 dt\right] < 2n^{-3/4}.$$

Consequently, according to Theorem 5, we have

$$r(U_n(f;x), f(x)) \leqslant \mu_f(8n^{-3/4}) + 8Mn^{-3/4},$$

where $\mu_f(\delta)$ is the modulus of nonmonotonicity of the function $f(x)$, and $M = \sup_x |f(x)|$.

From Theorem 5 one can also obtain a number of other estimates.

Mathematical Institute with Computing Center
Sofia, Bulgaria

Joint Institute for Nuclear Research

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Note: Figure translations are in progress. See original paper for figures.