ON A GENERALIZATION OF JORDAN THEORY

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Abstract

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MATHEMATICS

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ON A GENERALIZATION OF JORDAN THEORY

FOR A CERTAIN CLASS OF LINEAR OPERATORS

IN HILBERT SPACE

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It is known that the matrix of a linear operator B acting in a finite-dimensional complex linear space R can be reduced to the so-called Jordan normal form ([1]). Geometrically this means that the space R decomposes into a direct sum $R = R_1 + R_2 + \cdots + R_m$ of subspaces invariant with respect to the operator B. Moreover, the operator $B_i = B|R_i$ induced in the space R_i $(i = 1, 2, \dots, m)$, whose matrix is similar to a single Jordan block, has the property that the set of all its invariant subspaces is ordered by inclusion. The latter made it possible for M. S. Brodskii to generalize the concept of being single-block to the infinite-dimensional case ([2]).

Let $\mathfrak{H}$ be a separable Hilbert space and let A be a bounded linear operator acting in $\mathfrak{H}$, whose spectrum consists of a single point. The operator A is called **single-block** if one of any two of its invariant subspaces is contained in the other.

In the present note the principal result of Jordan theory is generalized to a certain class of infinite-dimensional operators.

We shall assign an operator A to the class Ω_0^+ if the following conditions are satisfied: 1) the spectrum of the operator A is concentrated at zero; 2) the imaginary component $A_I = \dfrac{1}{2i}(A - A^*)$ of the operator A is nonnegative; 3) $\operatorname{sp} A_I < \infty$ (see ([3])). Denote by $\sigma(A)$ the upper limit as $\lambda \to 0$ of the function $|\lambda| \ln \|(A - \lambda E)^{-1}\|$. If this limit is finite, then the number $\sigma(A)$ will be called the **type** of the operator A. For operators of the class Ω_0^+ the inequality

$$\sigma(A) \leqslant 2 \operatorname{sp} A_I, \tag{1}$$

always holds; moreover, for a simple operator $A \in \Omega_0^+$ to be single-block it is necessary and sufficient that equality hold in (1) ([2,4]).

We shall agree to say that the space $\mathfrak{H}$ is a **quasidirect sum** of some set $\mathfrak{M} = \{\mathfrak{H}_i\}$ of its subspaces if the following conditions are fulfilled: a) the linear span of the set $\mathfrak{M}$ is dense in $\mathfrak{H}$; b) if $\mathfrak{M}^{(k)}$ $(k = 1, 2)$ are arbitrary disjoint subsets of $\mathfrak{M}$, and $\mathfrak{H}^{(k)}$ is the closure of the linear span of the set $\mathfrak{M}^{(k)}$, then the subspaces $\mathfrak{H}^{(1)}$ and $\mathfrak{H}^{(2)}$ have zero intersection.

Let $\{\mathfrak{H}_i\}_{i=1}^m$ $(m \leqslant \infty)$ be a collection of invariant subspaces of a simple operator $A \in \Omega_0^+$, let $\mathfrak{H}_0$ be the smallest subspace containing all $\mathfrak{H}_i$ $(i = 1, 2, \dots, m)$, and let $A^i = A|\mathfrak{H}_i$ $(i = 0, 1, \dots, m)$.

Theorem 1. *In order that the subspace $\mathfrak{H}_0$ be the quasidirect sum of the set of subspaces $\{\mathfrak{H}_i\}_{i=1}^m$, it is necessary and sufficient that the equality*

$$\operatorname{sp} A_I^{(0)} = \sum_{i=1}^m \operatorname{sp} A_I^{(i)} \tag{2}$$

hold.

The following assertion may be regarded as an analogue of the theorem on the decomposition of a non-one-cell operator in a finite-dimensional space into a direct sum of one-cell operators.

Theorem 2. Let A be a simple operator of the class Ω_0^+, acting in the space $\mathfrak{H}$. Then the space $\mathfrak{H}$ decomposes into a quasi-direct sum of a finite or countable set $\{\mathfrak{H}_i\}_{i=1}^m$ $(1 \leq m \leq \infty)$ of subspaces invariant with respect to the operator A, satisfying the conditions:
1) the operator $A_i = A|_{\mathfrak{H}_i}$ $(i = 1, 2, \dots, m)$ is one-cell;
2) $\sigma(A) = \sigma(A_1) \geq \sigma(A_2) \geq \dots > 0$.

We note that the collection of subspaces $\{\mathfrak{H}_i\}$, as in the finite-dimensional case, is in general not determined uniquely. Therefore the question arises of invariants of the above-mentioned decomposition.

Theorem 3. Let $\{\mathfrak{H}_i\}_{i=1}^m$ and $\{\mathfrak{H}_i'\}_{i=1}^{m'}$ be two collections of subspaces invariant with respect to the operator $A \in \Omega_0^+$, satisfying the conditions of the preceding theorem,

$$A_i = A|_{\mathfrak{H}_i} \quad (i = 1, 2, \dots, m), \qquad A_i' = A|_{\mathfrak{H}_i'} \quad (i = 1, 2, \dots, m').$$

Then

$$m = m', \qquad \sigma(A_i) = \sigma(A_i') = \sigma_i \quad (i = 1, 2, \dots, m).$$

Let the subspace $\mathfrak{H}_0$ be invariant with respect to the operator $A \in \Omega_0^+$. Introduce the notation

$$\mu(\mathfrak{H}_0) = \operatorname{sp} A_I^{(0)} \qquad (A^{(0)} = A|_{\mathfrak{H}_0}).$$

For every number μ' in the interval $[0, \mu(\mathfrak{H}_0)]$ there exists ([5]) such an invariant subspace $\mathfrak{H}' \subseteq \mathfrak{H}_0$ of the operator A for which $\mu(\mathfrak{H}') = \mu'$. Using the notation of Theorem 3, for each $\sigma \in [0, \sigma_i]$ we find subspaces $\mathfrak{H}_i(\sigma)$ and $\mathfrak{H}_i'(\sigma)$, invariant with respect to the operator A, satisfying the conditions:

$$\sigma(A|_{\mathfrak{H}_i(\sigma)}) = 2\mu(\mathfrak{H}_i(\sigma)) = \sigma,$$

$$\sigma(A|_{\mathfrak{H}_i'(\sigma)}) = 2\mu(\mathfrak{H}_i'(\sigma)) = \sigma \qquad (i = 1, 2, \dots, m).$$

Next put

$$\mathfrak{H}(\sigma) = \overline{\mathfrak{H}_1(\sigma) + \cdots + \mathfrak{H}_i(\sigma) + \mathfrak{H}_{i+1} + \cdots} \qquad (\sigma_{i+1} < \sigma \leq \sigma_i),$$

$$\mathfrak{H}'(\sigma) = \overline{\mathfrak{H}_1'(\sigma) + \cdots + \mathfrak{H}_i'(\sigma) + \mathfrak{H}_{i+1}' + \cdots} \qquad (\sigma_{i+1} < \sigma \leq \sigma_i).$$

Theorem 4. The equality

$$\mathfrak{H}(\sigma) = \mathfrak{H}'(\sigma) \qquad (0 < \sigma \leq \sigma_1 = \sigma(A))$$

holds. In addition, $\mathfrak{H}(\sigma)$ is a maximal subspace invariant with respect to the operator A in which the type of the induced operator is equal to σ.

Let $E(\sigma)$ $(0 < \sigma \leq \sigma_1)$ be the orthoprojector in $\mathfrak{H}$ onto the subspace $\mathfrak{H}(\sigma)$, $E(0) = 0$. It is easy to see that $E(\sigma)$ is a spectral function of the operator A ([6]). The function $E(\sigma)$, determined in view of Theorem 4 by the operator A uniquely*, will be called the **canonical spectral function** of the operator A**.

If the operator A belongs to the class Ω_0^+, then the operator $B = -A^*$, obviously, also belongs to Ω_0^+. Let $\{\mathfrak{H}_i\}_{i=1}^m$ be a set of subspaces invariant with respect to the operator A, satisfying the conditions of Theorem 2. Consider the collection $\{\mathfrak{G}_i\}_{i=1}^m$ of subspaces defined by the equality

$$\mathfrak{G}_i = \mathfrak{H} \ominus \overline{\mathfrak{H}_1 + \cdots + \mathfrak{H}_{i-1} + \mathfrak{H}_{i+1} + \cdots} \qquad (i = 1, 2, \dots, m).$$

* If the operator A is not one-cell, then there corresponds to it an infinite set of different spectral functions.

** Yu. P. Ginzburg has recently ([7]) obtained a multiplicative representation of the characteristic function of an operator. As it turned out, in the case of an operator of the class Ω_0^+ this representation corresponds to the canonical spectral function of the operator.

Theorem 5. The space $\mathfrak{H}$ is the quasi-direct sum of the set of subspaces $\{\mathfrak{G}_i\}_{i=1}^m$. Moreover, each subspace $\mathfrak{G}_i$ is invariant with respect to the operator B, the operator induced in it, $B_i = B \mid \mathfrak{G}_i$, is one-cellular, and

$$\sigma(B_i) = \sigma(A_i) \qquad (i = 1, 2, \dots, m).$$

Theorems 3 and 4 contain answers to some of the questions posed to the author during a talk at the seminar on functional analysis under the direction of M. G. Krein in Odessa. The author takes this opportunity to express his sincere gratitude to all the participants of the seminar.

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Note: Figure translations are in progress. See original paper for figures.