ON A ONE-DIMENSIONAL ANALOGUE OF A SECOND-ORDER ELLIPTIC EQUATION DEGENERATING ON THE BOUNDARY

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ON A ONE-DIMENSIONAL ANALOGUE OF A SECOND-ORDER ELLIPTIC EQUATION DEGENERATING ON THE BOUNDARY

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In the paper [1], J. J. Kohn and L. Nirenberg posed the problem of finding solutions, smooth up to the boundary, of the equation

$$au'' + b_1 u' + cu = q, \qquad (1)$$

considered on the interval $(0, d)$. The coefficients $a(x), b_1(x), c(x)$ and the right-hand side $q(x)$ of equation (1) are assumed to be sufficiently smooth complex-valued functions on $(0, d)$, with $|a(x)| \neq 0$ for $x > 0$ and $a(0) = 0$. Equation (1) may be regarded as a one-dimensional analogue of a second-order elliptic equation degenerating on the boundary. The point $x = 0$ then plays the role of the "boundary of degeneration." The smoothness conditions for solutions of elliptic-parabolic equations obtained in [1, 2] are, of course, applicable also to equation (1). However, the method used in those papers is connected with the necessity of continuing equation (1) to $[-\varepsilon, 0]$ ($\varepsilon > 0$) while preserving certain properties, which substantially narrows the class of equations considered. In the present note, differentiability conditions are given for solutions of equation (1) that are not connected with the method of continuing the equation. At the same time, we are able to reduce by one derivative the smoothness conditions imposed on the right-hand side of equation (1).

By inverting the operator $au'' + b_1 u'$, equation (1) can be reduced to one of the following integral equations:

$$u(x) - \int_0^d \mathcal{K}(x, r) u(r)\, dr = \int_0^d k(x, r) q(r)\, dr + \mu \int_x^d e^{P_1(s)}\, ds + \nu; \qquad (2)$$

$$u(x) - \int_0^d \mathcal{L}(x, r) u(r)\, dr = \int_0^d l(x, r) q(r)\, dr + \mu \int_x^d e^{P_1(s)}\, ds + \nu, \qquad (3)$$

where

$$k(x,r)=\begin{cases}0, & r\le x,\\ \displaystyle\int_x^r \frac{e^{P_1(s)-P_1(r)}}{a(r)}\,ds, & r>x;\end{cases}$$

$$l(x,r)=\begin{cases}\displaystyle\int_r^x \frac{e^{P_1(s)-P_1(r)}}{a(r)}\,ds, & r<x,\\ 0, & r\ge x;\end{cases}$$

$$\mathcal{K}(x,r)=-k(x,r)c(r);\qquad \mathcal{L}(x,r)=-l(x,r)c(r);\qquad P_1(s)=\int_s^d \frac{b_1(y)}{a(y)}\,dy;$$

μ and ν are arbitrary constants.

Let

$$b_m(x)=b_1(x)+(m-1)a'(x);\qquad \beta_m(x)=\operatorname{Re}\frac{b_1(x)\overline{a}(x)}{|a(x)|}+(m-1)|a(x)|\quad (0<x<d,\ m=0,1,2,...).$$

The integral operators

$$KQ=\int_0^d \mathcal{K}(x,r)Q(r)\,dr\quad\text{and}\quad LQ=\int_0^d \mathcal{L}(x,r)Q(r)\,dr$$

are Volterra operators, and therefore, in order to prove the next two lemmas, it is sufficient to prove the boundedness of the kernels $\mathcal{K}(x,r)$, $\mathcal{L}(x,r)$.

Lemma 1. Let $q\in\mathcal{L}_1(0,d)$; $b_1,c\in C^{(0)}[0,d]$; $a(x)$ be a function differentiable for $x>0$; $|a(x)|\ne 0$ for $x>0$; $a(0)=0$. If

$$\lim_{x\to 0}\beta_0(x)<0,$$

then the series

$$u(x)=\int_0^d k(x,r)q(r)\,dr+\mu\int_x^d e^{P_1(s)}\,ds+\nu+$$
$$+\sum_{n=1}^{\infty}K^n\left(\int_0^d k(r,s)q(s)\,ds+\mu\int_r^d e^{P_1(s)}\,ds+\nu\right)\tag{4}$$

converges absolutely and uniformly on $[0,d]$, and its sum $u(x)$ is a continuous on $[0,d]$ solution of equation (2) for arbitrary constants μ and ν.

Lemma 2. Let q, b_1, c, a satisfy the conditions of Lemma 1. If

$$\lim_{x\to+0}\beta_0(x) > 0,$$

then the series

$$u(x) = \int_0^d l(x,r)q(r)\,dr + \nu + \sum_{n=1}^{\infty} L^n \left(\int_0^d l(r,s)q(s)\,ds + \nu\right) \tag{5}$$

converges absolutely and uniformly on $[0,d]$, and its sum $u(x)$ is a continuous on $[0,d]$ solution of equation (3) for any constant ν.

Remark. If the conditions of Lemma 2 are satisfied and $a \in C^{(1)}[0,d]$, the function $e^{P_1(x)}$ does not belong to $\mathcal{L}_1(0,d)$. Since we are interested only in solutions bounded on $[0,d]$, in (3) one should set $\mu = 0$.

To study the differential properties of the solutions (4) and (5) constructed, we shall study the differential properties of the corresponding integral operators.

Lemma 3. Let m be a positive integer; $b_1 \in C^{(2m-2)}[0,d]$; $c \in C^{(m-1)}[0,d]$; $a \in C^{(2m-2)} \cap C^{(1)}[0,d]$; $|a(x)| \neq 0$ for $x > 0$; $a(0) = 0$; $Q \in C^{(m-1)}[0,d]$; $Q(d) = Q'(d) = \dots = Q^{(m-2)}(d) = 0$.

If, for all $x \in [0,d]$, the inequalities

$$\beta_0(x) < 0, \qquad \beta_t(x) - |a(x)|\,\mathrm{Re}\sum_{i=1}^{t-1} R_i'(x) < 0 \quad (t = 1,2,\dots,m), \tag{6}$$

hold, where $R_t(x)$ are determined successively from the relations

$$R_1(x) = \ln[-b_1(x)], \qquad R_t(x) = \ln\left[-b_t(x) + a(x)\sum_{i=1}^{t-1} R_i'(x)\right]$$

$$(t = 2,3,\dots,m-1),$$

then the function $v(x) = KQ$ has continuous derivatives on $[0,d]$ up to order m, and the formula

$$\frac{d^t v}{dx^t} = -\int_x^d \frac{e^{P_t(x)-P_t(r)}}{a(r)}\,T_{0,t}(r)Q^{(t-1)}(r)\,dr +$$

$$+\sum_{j=1}^{t-1}\int_x^d \left[\int_x^r \frac{e^{P_t(x)-P_t(s)}}{a(s)}\,T_{j,t}(s)\,ds\right] Q^{(t-j)}(r)\,dr,$$

where

$$P_t(x) = \int_x^d \frac{b_t(y)}{a(y)}\,dy + \sum_{i=1}^{t-1} R_i(x);$$

$$T_{j,t+1}(x) = T_{j,t}(x) - R_t'(x)T_{j-1,t}(x) + T_{j-1,t}'(x); \tag{7}$$

$$T_{-1,t} \equiv 0; \qquad T_{j,t} \equiv 0 \text{ for } j \geq t; \qquad T_{0,1}(x) = -c(x); \qquad t = 1, 2, \dots, m.$$

Lemma 4. Let m be a positive integer, $b_1 \in C^{(m-1)}[0,d]$, $c \in C^{(m-1)}[0,d]$, $a \in C^{(m-1)} \cap C^{(1)}[0,d]$, $|a(x)| > 0$ for $x > 0$; $a(0) = 0$; $Q \in C^{(m-1)}[0,d]$, $Q(0) = Q^{(1)}(0) = \dots = Q^{(m-2)}(0) = 0$.

If for all $x \in [0,d]$ the inequalities

$$\beta_0(x) > 0; \qquad \beta_t(x) - |a(x)| \operatorname{Re} \sum_{i=1}^{t-1} \Phi_i'(x) > 0, \qquad t = 1, 2, \dots, m, \tag{8}$$

are satisfied, where $\Phi_t(x)$ are successively defined from the relations

$$\Phi_1(x) = \ln b_1(x); \qquad \Phi_t(x) = \ln \left[b_t(x) - a(x) \sum_{i=1}^{t-1} \Phi_i'(x) \right], \qquad t = 2, 3, \dots, m-1,$$

then the function $w(x) = LQ$ has continuous derivatives on $[0,d]$ up to order m, and the formula

$$\frac{d^t w}{dx^t} = \int_0^x e^{P_t(x)-P_t(r)} \frac{T_{0,t}(r)}{a(r)} Q^{(t-1)}(r)\, dr + \sum_{j=1}^{t-1} \int_0^x \left[\int_r^x e^{P_t(x)-P_t(s)} \frac{T_{j,t}(s)}{a(s)}\, ds \right] Q^{(t-j)}(r)\, dr,$$

holds, where $P_t(x)$ and $T_{j,t}(x)$ are defined by the relations (7) with $R_i(x)$ replaced by $\Phi_i(x)$.

Remark. If equations (2) and (3) are considered for $d = \delta$, where δ is a sufficiently small positive number, then condition (6) in Lemma 3 (respectively, condition (8) in Lemma 4) may be replaced by the conditions $\beta_t(x) < 0$ ($\beta_t(x) > 0$) for $x \in [0,\delta]$ and $t = 0, 1, \dots, m$.

With the aid of Lemmas 3 and 4, the following two lemmas are easily obtained.

Lemma 5. Let m be a positive integer; $q \in C^{(m-1)}[0,d]$; $b_1 \in C^{(m-1)}[0,d]$; $c \in C^{(m-1)}[0,d]$, $a \in C^{(m-1)} \cap C^{(1)}[0,d]$, $a(0) = 0$; $|a(x)| > 0$ for $x > 0$; $d_0 \leq \min\{\delta, d\}$, where δ is a sufficiently small positive constant depending only on $a(x)$ and $b_1(x)$.

If

$$\lim_{x\to+0}\beta_m(x) < 0,$$

then every solution of equation (2) that vanishes together with its derivative at the point $x = d_0$ belongs to $C^{(m)}[0, d_0]$, and for it the estimate

$$\|u\|_{C^{(m)}[0,d_0]} \leq c_1 \left(\|q\|_{C^{(m-1)}[0,d_0]} + \|u\|_{C^{(0)}[0,d_0]}\right), \tag{9}$$

holds, where the constant $c_1 > 0$ depends only on m, $a(x)$, $b_1(x)$, $c(x)$, and their derivatives on $[0, d_0]$.

Corollary 1. The solution (4) of equation (2) for $\nu = \mu = 0$, under the hypotheses of Lemma 5, belongs to $C^{(m)}[0, d_0]$, and for it the estimate

$$\|u\|_{C^{(m)}[0,d_0]} \leq c_2\|q\|_{C^{(m-1)}[0,d_0]}. \tag{10}$$

is valid.

Lemma 6. Let $a(x)$, $b_1(x)$, $c(x)$ have the same smoothness as in Lemma 5, and let $d_0 \leq \min\{\delta, d\}$, where δ is a sufficiently small positive constant depending only on $a(x)$, $b_1(x)$.

If

$$\lim_{x\to+0}\beta_0(x) > 0$$

and

$$q(0) = q^{(1)}(0) = ... = q^{(m-2)}(0) = 0, \tag{11}$$

then every solution of equation (3), bounded on $[0, d_0]$ and vanishing

to zero at $x = 0$, belongs to $C^{(m)}[0, d_0]$, and for it the estimate (9) is valid.

Corollary 2. The solution (5) of equation (3) for $\nu = 0$ and under the conditions of Lemma 6 belongs to $C^{(m)}[0, d_0]$, and for it the estimate (10) is valid.

Corollary 3. Suppose that all the conditions of Lemma 6 are satisfied, except for condition (11). Then any bounded on $[0, d_0]$ solution of equation (1) belongs to $C^{(m)}[0, d_0]$, and for it the estimate (9) is valid.

The proof of Corollary 2 is based on the possibility of constructing a polynomial $\chi(x)$ ($\chi(0) = 0$) of degree $m - 1$, which satisfies the conditions

$$(a\chi'' + b_1\chi' + c\chi)^{(t)}_{x=0} = q^{(t)}(0), \qquad t = 0, 1, \dots, m-2.$$

After this, the solution of equation (1) for arbitrary $q \in C^{(m-1)}[0,d]$ is reduced to the solution of the same equation, but with a right-hand side satisfying conditions (11).

With the aid of Lemma 5 one proves:

Theorem 1. Let m be a positive integer; $q(x)$, $c(x) \in C^{(m-1)}[0,d]$; $a(x)$, $b_1(x) \in C^{(m-1)} \cap C^{(1)}[0,d]$; $a(0) = 0$; $|a(x)| > 0$ for $x > 0$.

If

$$\overline{\lim_{x \to +0}}\, \beta_m(x) < 0,$$

then every solution of equation (1) belongs to $C^{(m)}[0,d]$, and for it the estimate is valid

$$\|u\|_{C^{(m)}[0,d]} \leq c_3 \big(\|q\|_{C^{(m-1)}[0,d]} + \|u\|_{C^{(0)}[0,d]} \big). \tag{12}$$

The constant $c_3 > 0$ depends on m, $a(x)$, $b_1(x)$, $c(x)$, and their derivatives on $[0,d]$.

Using Corollary 2, one can prove the following theorem.

Theorem 2. Let m be a positive integer and let $a(x)$, $b_1(x)$, $c(x)$, $q(x)$ satisfy the conditions of Theorem 1.

If

$$\underline{\lim_{x \to +0}}\, \beta_0(x) > 0,$$

then every bounded on $[0,d]$ solution of equation (1) belongs to $C^{(m)}[0,d]$, and for it the estimate (12) is valid.

Theorems 1 and 2 can be generalized to the case of the spaces $C^{(m+\alpha)}[0,d]$ $(0 < \alpha < 1)$.

The example given in the paper of J. J. Kohn and L. Nirenberg (([1], p. 489)) shows that the condition

$$\overline{\lim_{x \to +0}}\, \beta_m(x) < 0$$

in Theorem 1 is sharp, since one can specify a bounded on $[0,d]$ solution of the homogeneous equation (1) which, when the condition

$$\overline{\lim_{x \to +0}}\, \beta_m(x) < 0$$

is satisfied, has m continuous derivatives but does not have an $(m+1)$-st derivative at the point $x = 0$.

Let us also consider the equation $xu'' + b_1 u' + cu = x^\alpha$ $(0 < \alpha < 1,\ x \in (0,\delta))$, where the constant $b_1 < -1$ (or $b_1 > 1$). With the aid of the series (4) (respectively (5)) it is easy to construct a bounded on $[0,\delta]$ solution of this equation which has no second derivative at the point $x = 0$. Hence it follows that the condition $q \in C^{(m-1)}[0, \delta]$ in Theorems 1 and 2 cannot be weakened.

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CITED LITERATURE

1. J. J. Kohn, L. Nirenberg, Comm. Pure and Appl. Math., **18**, No. 3, 443 (1965).

2. O. A. Oleinik, DAN, **163**, No. 3 (1965).

Note: Figure translations are in progress. See original paper for figures.