An elliptic system of partial differential equations with a singular point at the origin

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Date: 1967-01-01T00:00:00+00:00

Abstract

Full Text

ON AN ELLIPTIC SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS WITH A SINGULAR POINT AT THE ORIGIN

The analytical theory of the Briot-Bouquet equation is well known:

$$\frac{dy}{dx} = f(x, y)$$

where $f(x, y)$ is a function holomorphic in the neighborhood of $x = y = 0$. The purpose of this note is to demonstrate that certain analogous results hold for the equation (here and hereafter, symbols with a bar denote complex conjugate quantities):

$$\bar{z}\frac{\partial u}{\partial \bar{z}} = f(z, \bar{z}, u, \bar{u}) \tag{1}$$

where $f(z, \bar{z}, u, \bar{u})$ is a holomorphic function in the neighborhood of $(0, 0, 0, 0)$ such that $f(0, 0, 0, 0) = 0$. According to [?], $\frac{\partial}{\partial \bar{z}}$ is the operator derivative, or according to [?], the generalized derivative. Therefore, equation (1) can be rewritten in the form:

$$\bar{z}\frac{\partial u}{\partial \bar{z}} = f(z, \bar{z}, u, \bar{u}) \tag{2}$$

Linear homogeneous equations of the form (2) have been previously investigated in [?, ?, ?]. In this study, we first address the question of the existence of a solution to equation (2) that is holomorphic with respect to the powers of z and $\bar{z}$ and vanishes at $z = \bar{z} = 0$. Let

$$f(z, \bar{z}, u, \bar{u}) = \sum f_{pqkl} z^p \bar{z}^q u^k \bar{u}^l$$

If the specified solution to equation (2) exists, it can be represented in the form:

$$u = \sum_{m+n=1}^{\infty} a_{mn} z^m \bar{z}^n \tag{3}$$

To determine the coefficients, we have the following equations:

$$\begin{aligned} (n-a)a_{mn} - b\bar{a}_{nm} &= \Phi_{mn} \\ -ba_{mn} + (m-a)\bar{a}_{nm} &= \Phi_{nm} \end{aligned}$$

The terms Φ_{mn} and Φ_{nm} are polynomials with respect to those coefficients $a_{m'n'}$ for which $m' + n' < m + n$, $m' \leq m$, and $n' \leq n$. These polynomials are obtained as the coefficients of the series resulting from the substitution of series (3) into the expression $F(z, \bar{z}, u, \bar{u}) - au - b\bar{u}$. We observe that if the determinants of the systems $\Delta_{mn} = (n-a)(m-a) - b^2$ are non-zero for all non-negative integers m and n satisfying the inequality $m + n > 0$, then a formal solution (3) exists and is unique.

We shall now prove the convergence of series (3) under the assumption that all $\Delta_{mn} \neq 0$. It is easy to see that there exists a positive constant B such that we have

$$\frac{m}{|\Delta_{mn}|} < B, \quad \frac{n}{|\Delta_{mn}|} < B, \quad \frac{1}{|\Delta_{mn}|} < B$$

for all non-negative integers $m + n > 0$. Suppose there exists a majorant for the series $F(z, \bar{z}, u, \bar{u})$. Let us then consider the equation $U = BF(z, \bar{z}, U, \bar{U})$. It is easily seen that this equation possesses a holomorphic solution where all coefficients are positive. To determine these coefficients, we obtain the following equations:

$$A_{mn} = B\Phi_{mn} + B\Phi_{nm}$$

These are polynomials with respect to those indices for which $m' < m$ and $n' < n$. From the convergence of the majorizing series, the convergence of series (3) follows. Thus, we obtain the following:

Theorem

If $(m-a)(n-a)-b^2 \neq 0$ for all non-negative integers m, n such that $m+n>0$, the equation has a unique holomorphic solution in the neighborhood of the origin.

Suppose now that for certain values of m and n, the expression vanishes. If a is a real number and $b=0$, then $n-a$ can be zero for at most one value. If a is a real irrational number, the expression can vanish for only one pair of numbers. Now, let a be a rational number. Then the expression will vanish for an infinite number of pairs (m, n) if a is equal to a non-negative integer. In all other cases, it can vanish only for a finite set of values.

If the determinant vanishes at (m_0, n_0) and is non-zero for all other values, and if the condition $(m_0-a)a_{m_0 n_0} - b\bar{a}_{n_0 m_0} = \Phi_{m_0 n_0}$ is not satisfied, then equation (2) does not possess a solution of the specified form. If the condition is satisfied, the solution depends on an arbitrary constant.

Regarding the case where multiple pairs of indices cause the determinant to vanish, say $(m_1, n_1), \ldots, (m_k, n_k)$, the equation will have a solution (3) if and only if certain consistency conditions are met. If these conditions hold identically with respect to the constants $C_1, \ldots, C_k$, then the solution depends on these arbitrary constants.

Finally, let $b=0$ and $a=s$, where s is a non-negative integer. For the existence of a formal solution (3) to equation (2) in this case, it is necessary and sufficient that $\Phi_{mn}=0$ for the corresponding indices. If these conditions are satisfied, the solution (3) will depend on an infinite number of arbitrary parameters, and the question of convergence depends on the choice of these parameters.

We now consider equation (2) under the assumption that $f(0,0,0,0)=0$. We seek a solution in the form:

$$u = \sum c_{mnjk} z^m u^n \bar{z}^j \bar{u}^k \tag{15}$$

To determine the coefficients, we obtain the corresponding equations:

$$\begin{gathered}(a-\alpha)p_{0001} - bp_{0010} = 0\\ -bp_{0001} + (\beta-\alpha)p_{0010} = 0\end{gathered}$$

If the determinant of this system is zero, we can take $p_{0010}=C_1$ and $p_{0001}=C_2$ as arbitrary constants. To determine the other coefficients p_{mnjk}, we obtain:

$$(n + j\beta + k\alpha - a)p_{mnjk} - b\bar{p}_{mkjn} = R_{mnjk}$$

If $j\beta + k\alpha - n\alpha \neq 0$ for all m, n, j, k such that $m+n+j+k \geq 1$ and $j+k>0$, then all coefficients p_{mnjk} are uniquely determined. Under these conditions, the equation possesses a unique formal solution (15) that depends on two arbitrary constants. The convergence of this series for sufficiently small variables follows from the existence of a majorizing series.

Theorem

Let κ_{mnjk} be defined such that the denominators in the coefficient formulas do not vanish. If $f(z, \bar{z}, 0, 0) \equiv 0$, equation (2) possesses a solution in the form of a convergent series (15) for sufficiently small $|z|$. The coefficients p_{0010} and p_{0001} are determined by the initial linear system.

In the case where $b = 0$, the equation under the condition $f(z, \bar{z}, 0, 0) \equiv 0$ possesses a convergent solution for sufficiently small values of the variables. This solution depends on the parameters C_1, C_2 as defined by the simplified linear relations.

For the general case with q variables, the equation possesses a unique formal solution depending on $2q$ arbitrary constants $C_1, \dots, C_{2q}$. The convergence of this series for sufficiently small $|z_1|, \dots, |z_q|$ is guaranteed provided that the eigenvalues of the linear part satisfy the required non-resonance conditions.

Theorem

Suppose that there exists a constant $B > 0$ such that the small divisor conditions are satisfied for all non-negative integers $m, n, p, q, \dots$. Then, assuming $f(z, \bar{z}, 0, 0) = 0$, equation (2) possesses a convergent solution for sufficiently small $|z|$ and $|\bar{z}|$. The arbitrary constants can always be chosen such that the convergence inequalities are satisfied. This result generalizes the previous theorems to higher-dimensional systems.

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Received by the editorial office on January 28, 1966.

Figures

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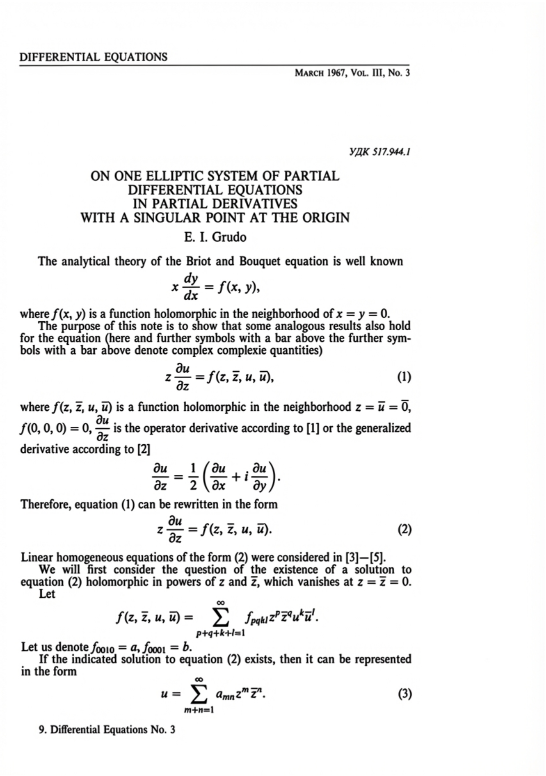


DIFFERENTIAL EQUATIONS
MARCH 1967, VOL. III, No. 3

УДК 517.944.1

ON ONE ELLIPTIC SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS IN PARTIAL DERIVATIVES WITH A SINGULAR POINT AT THE ORIGIN

E. I. Grudo

The analytical theory of the Briot and Bouquet equation is well known

$$x\frac{dy}{dx} = f(x, y),$$

where $f(x, y)$ is a function holomorphic in the neighborhood of $x = y = 0$.

The purpose of this note is to show that some analogous results also hold for the equation (here and further symbols with a bar above the further symbols with a bar above denote complex complexie quantities)

$$z\frac{\partial u}{\partial z} = f(z, \overline{z}, u, \overline{u}), \tag{1}$$

where $f(z, \overline{z}, u, \overline{u})$ is a function holomorphic in the neighborhood $z = \overline{u} = \overline{0}$, $f(0, 0, 0) = 0$, $\frac{\partial u}{\partial z}$ is the operator derivative according to [1] or the generalized derivative according to [2]

$$\frac{\partial u}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right).$$

Therefore, equation (1) can be rewritten in the form

$$z\frac{\partial u}{\partial z} = f(z, \overline{z}, u, \overline{u}). \tag{2}$$

Linear homogeneous equations of the form (2) were considered in [3]—[5].

We will first consider the question of the existence of a solution to equation (2) holomorphic in powers of z and $\overline{z}$, which vanishes at $z = \overline{z} = 0$.

Let

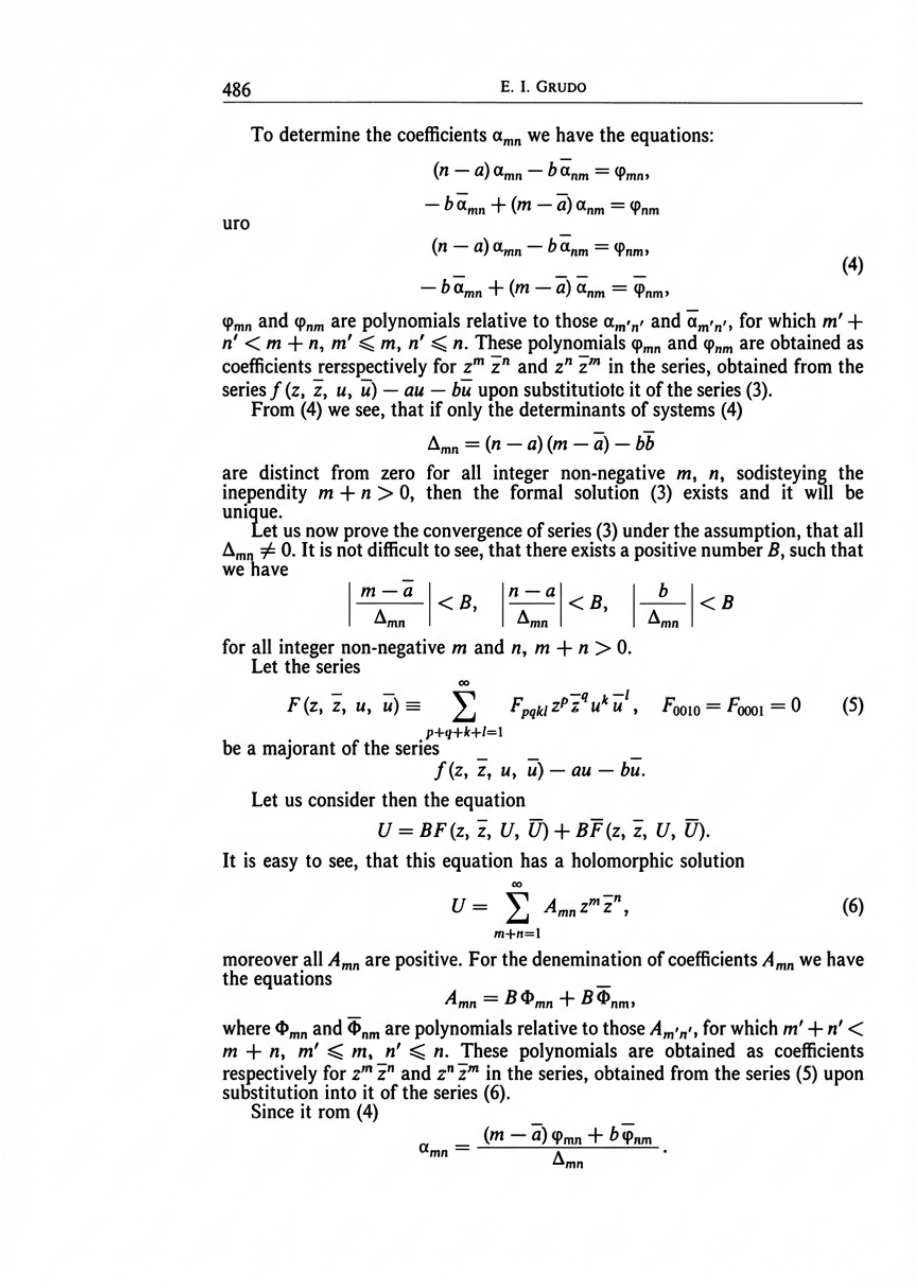
$$f(z, \overline{z}, u, \overline{u}) = \sum_{p+q+k+l=1}^{\infty} f_{pqkl} z^p \overline{z}^q u^k \overline{u}^l.$$

Let us denote $f_{0010} = a$, $f_{0001} = b$.

If the indicated solution to equation (2) exists, then it can be represented in the form

$$u = \sum_{m+n=1}^{\infty} a_{mn} z^m \overline{z}^n. \tag{3}$$

Figure 1: Figure 1



To determine the coefficients α_{mn} we have the equations:

$$(n-a)\,\alpha_{mn} - b\,\bar{\alpha}_{nm} = \varphi_{mn},$$

$$-\,b\,\bar{\alpha}_{mn} + (m-\bar{a})\,\alpha_{nm} = \varphi_{nm}$$

uro

$$\begin{aligned}(n-a)\,\alpha_{mn} - b\,\bar{\alpha}_{nm} &= \varphi_{nm},\\ -\,b\,\bar{\alpha}_{mn} + (m-\bar{a})\,\bar{\alpha}_{nm} &= \bar{\varphi}_{nm},\end{aligned} \tag{4}$$

φ_{mn} and φ_{nm} are polynomials relative to those $\alpha_{m'n'}$ and $\bar{\alpha}_{m'n'}$, for which $m' + n' < m+n$, $m' \leqslant m$, $n' \leqslant n$. These polynomials φ_{mn} and φ_{nm} are obtained as coefficients rersspectively for $z^m\,\bar{z}^n$ and $z^n\,\bar{z}^m$ in the series, obtained from the series $f(z,\,\bar{z},\,u,\,\bar{u}) - au - b\bar{u}$ upon substitutiotc it of the series (3).

From (4) we see, that if only the determinants of systems (4)

$$\Delta_{mn} = (n-a)\,(m-\bar{a}) - b\bar{b}$$

are distinct from zero for all integer non-negative m, n, sodisteying the inependity $m+n>0$, then the formal solution (3) exists and it will be unique.

Let us now prove the convergence of series (3) under the assumption, that all $\Delta_{mn} \neq 0$. It is not difficult to see, that there exists a positive number B, such that we have

$$\left|\frac{m-\bar{a}}{\Delta_{mn}}\right| < B, \quad \left|\frac{n-a}{\Delta_{mn}}\right| < B, \quad \left|\frac{b}{\Delta_{mn}}\right| < B$$

for all integer non-negative m and n, $m+n>0$.

Let the series

$$F(z,\,\bar{z},\,u,\,\bar{u}) \equiv \sum_{p+q+k+l=1}^{\infty} F_{pqkl}\, z^p\,\bar{z}^q\,u^k\,\bar{u}^l, \quad F_{0010} = F_{0001} = 0 \tag{5}$$

be a majorant of the series

$$f(z,\,\bar{z},\,u,\,\bar{u}) - au - b\bar{u}.$$

Let us consider then the equation

$$U = BF(z,\,\bar{z},\,U,\,\bar{U}) + B\bar{F}(z,\,\bar{z},\,U,\,\bar{U}).$$

It is easy to see, that this equation has a holomorphic solution

$$U = \sum_{m+n=1}^{\infty} A_{mn}\, z^m\,\bar{z}^n, \tag{6}$$

moreover all A_{mn} are positive. For the denemination of coefficients A_{mn} we have the equations

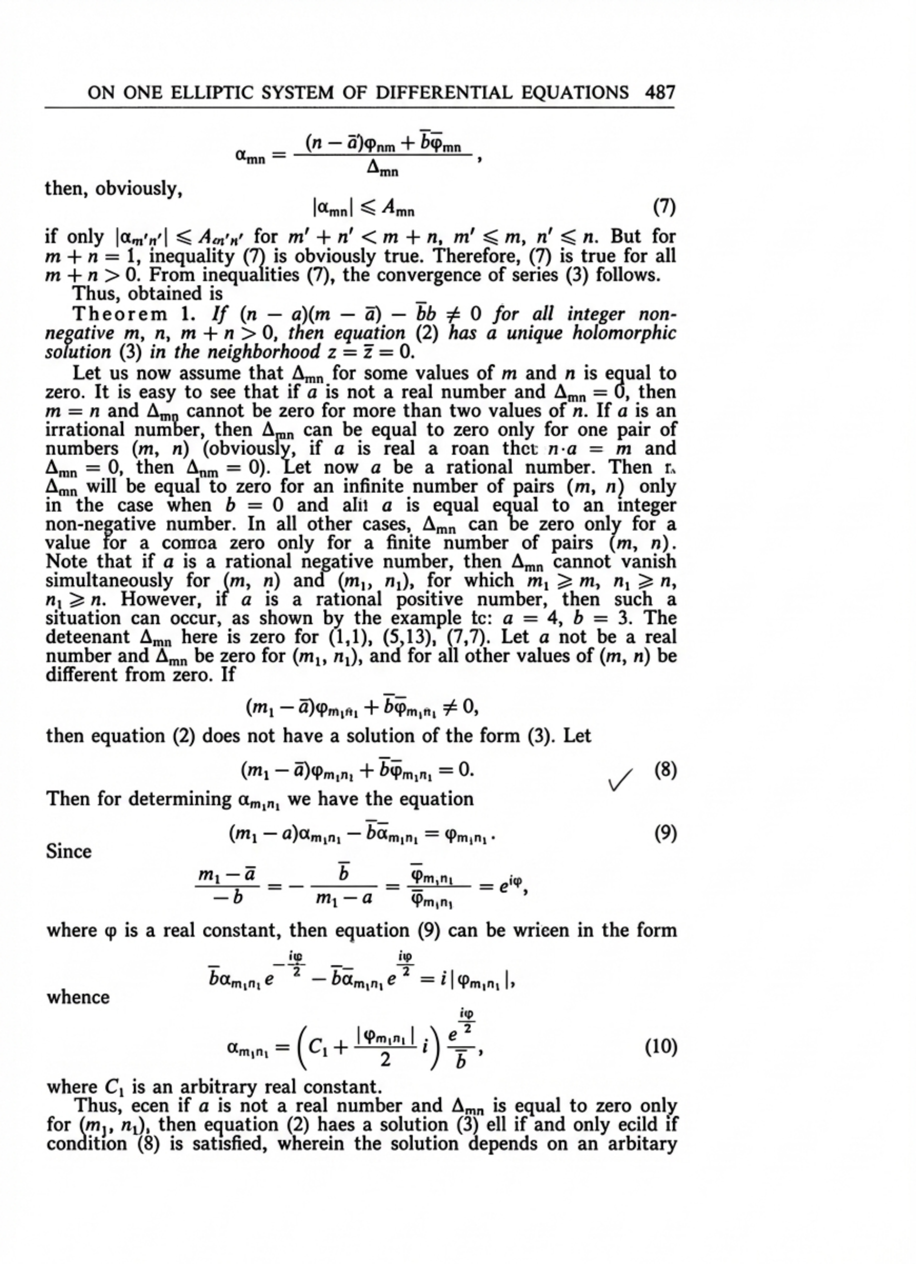
$$A_{mn} = B\,\Phi_{mn} + B\,\bar{\Phi}_{nm},$$

where Φ_{mn} and $\bar{\Phi}_{nm}$ are polynomials relative to those $A_{m'n'}$, for which $m'+n' < m+n$, $m' \leqslant m$, $n' \leqslant n$. These polynomials are obtained as coefficients respectively for $z^m\,\bar{z}^n$ and $z^n\,\bar{z}^m$ in the series, obtained from the series (5) upon substitution into it of the series (6).

Since it rom (4)

$$\alpha_{mn} = \frac{(m-\bar{a})\,\varphi_{mn} + b\,\bar{\varphi}_{nm}}{\Delta_{mn}}.$$

Figure 2: Figure 2



$$\alpha_{mn} = \frac{(n-\bar{a})\varphi_{nm} + \bar{b}\bar{\varphi}_{mn}}{\Delta_{mn}},$$

then, obviously,

$$|\alpha_{mn}| \leqslant A_{mn} \qquad (7)$$

if only $|\alpha_{m'n'}| \leqslant A_{m'n'}$ for $m' + n' < m + n$, $m' \leqslant m$, $n' \leqslant n$. But for $m + n = 1$, inequality (7) is obviously true. Therefore, (7) is true for all $m + n > 0$. From inequalities (7), the convergence of series (3) follows.

Thus, obtained is

Theorem 1. *If* $(n - a)(m - \bar{a}) - \bar{b}b \neq 0$ *for all integer non-negative* m, n, $m + n > 0$, *then equation* (2) *has a unique holomorphic solution* (3) *in the neighborhood* $z = \bar{z} = 0$.

Let us now assume that Δ_{mn} for some values of m and n is equal to zero. It is easy to see that if a is not a real number and $\Delta_{mn} = 0$, then $m = n$ and Δ_{mn} cannot be zero for more than two values of n. If a is an irrational number, then Δ_{mn} can be equal to zero only for one pair of numbers (m, n) (obviously, if a is real a roan thct $n\cdot a = m$ and $\Delta_{mn} = 0$, then $\Delta_{nm} = 0$). Let now a be a rational number. Then r. Δ_{mn} will be equal to zero for an infinite number of pairs (m, n) only in the case when $b = 0$ and alıı a is equal equal to an integer non-negative number. In all other cases, Δ_{mn} can be zero only for a value for a comoa zero only for a finite number of pairs (m, n). Note that if a is a rational negative number, then Δ_{mn} cannot vanish simultaneously for (m, n) and (m_1, n_1), for which $m_1 \geqslant m$, $n_1 \geqslant n$, $n_1 \geqslant n$. However, if a is a rational positive number, then such a situation can occur, as shown by the example tc: $a = 4$, $b = 3$. The deteenant Δ_{mn} here is zero for (1,1), (5,13), (7,7). Let a not be a real number and Δ_{mn} be zero for (m_1, n_1), and for all other values of (m, n) be different from zero. If

$$(m_1 - \bar{a})\varphi_{m_1n_1} + \bar{b}\bar{\varphi}_{m_1n_1} \neq 0,$$

then equation (2) does not have a solution of the form (3). Let

$$(m_1 - \bar{a})\varphi_{m_1n_1} + \bar{b}\bar{\varphi}_{m_1n_1} = 0. \qquad \checkmark \qquad (8)$$

Then for determining $\alpha_{m_1n_1}$ we have the equation

$$(m_1 - a)\alpha_{m_1n_1} - \bar{b}\bar{\alpha}_{m_1n_1} = \varphi_{m_1n_1}. \qquad (9)$$

Since

$$\frac{m_1 - \bar{a}}{-b} = -\frac{\bar{b}}{m_1 - a} = \frac{\bar{\varphi}_{m_1n_1}}{\bar{\varphi}_{m_1n_1}} = e^{i\varphi},$$

where φ is a real constant, then equation (9) can be wrieen in the form

$$\bar{b}\alpha_{m_1n_1} e^{-\frac{i\varphi}{2}} - \bar{b}\bar{\alpha}_{m_1n_1} e^{\frac{i\varphi}{2}} = i\,|\varphi_{m_1n_1}|,$$

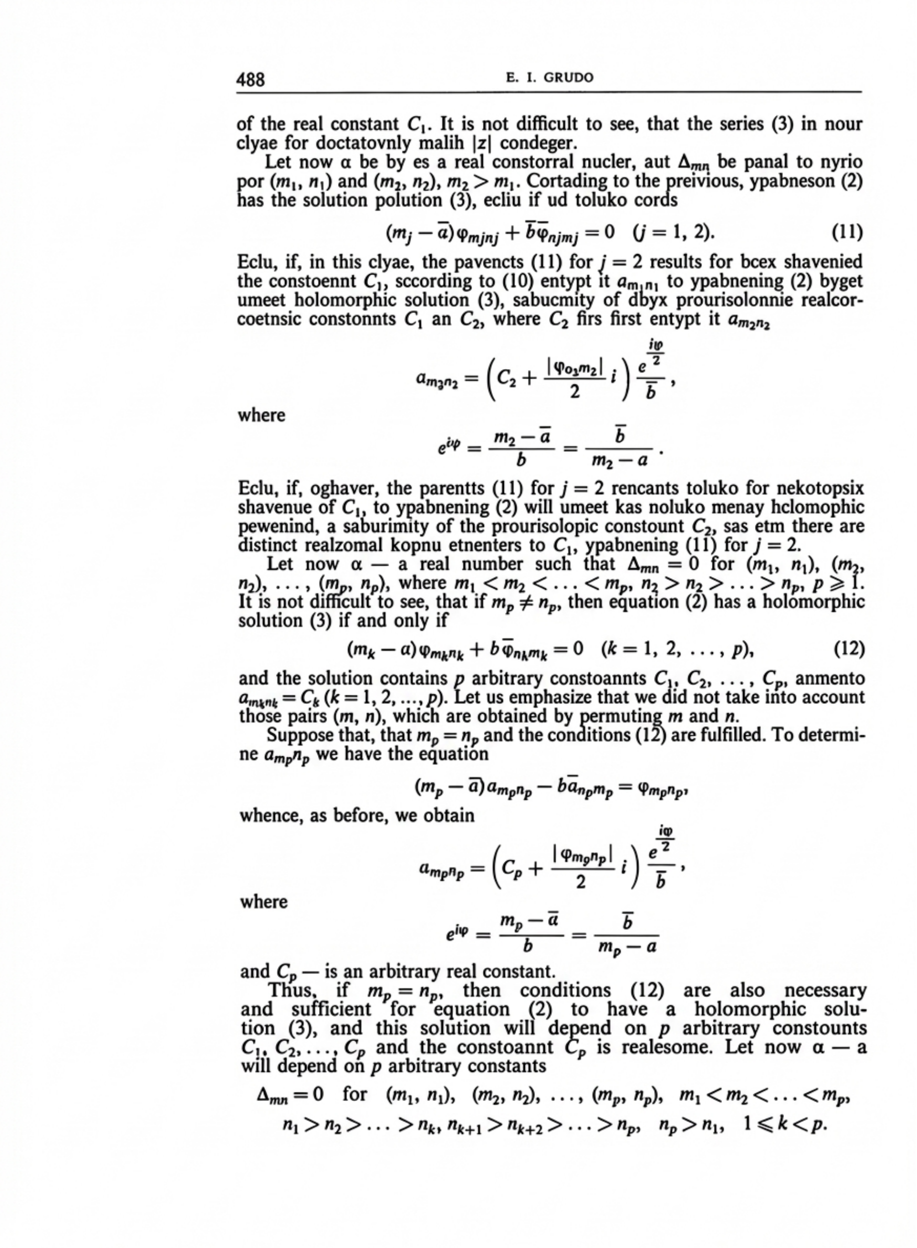
whence

$$\alpha_{m_1n_1} = \left(C_1 + \frac{|\varphi_{m_1n_1}|}{2}\, i\right)\frac{e^{\frac{i\varphi}{2}}}{\bar{b}}, \qquad (10)$$

where C_1 is an arbitrary real constant.

Thus, ecen if a is not a real number and Δ_{mn} is equal to zero only for (m_1, n_1), then equation (2) haes a solution (3) ell if and only ecild if condition (8) is satisfied, wherein the solution depends on an arbitary

Figure 3: Figure 3



of the real constant C_1. It is not difficult to see, that the series (3) in nour clyae for doctatovnly malih $|z|$ condeger.

Let now α be by es a real constorral nucler, aut Δ_{mn} be panal to nyrio por (m_1, n_1) and (m_2, n_2), $m_2 > m_1$. Cortading to the preivious, ypabneson (2) has the solution polution (3), ecliu if ud toluko cords

$$(m_j - \overline{a})\varphi_{mjnj} + \overline{b}\overline{\varphi}_{njmj} = 0 \quad (j = 1, 2). \tag{11}$$

Eclu, if, in this clyae, the pavencts (11) for $j = 2$ results for bcex shavenied the constoennt C_1, sccording to (10) entypt it $a_{m_1n_1}$ to ypabnening (2) byget umeet holomorphic solution (3), sabucmity of dbyx prourisolonnie realcorcoetnsic constonnts C_1 an C_2, where C_2 firs first entypt it $a_{m_2n_2}$

$$a_{m_3n_2} = \left(C_2 + \frac{|\varphi_{o_3m_2}|}{2}\, i\right)\frac{e^{\frac{i\varphi}{2}}}{\overline{b}},$$

where

$$e^{i\varphi} = \frac{m_2 - \overline{a}}{b} = \frac{\overline{b}}{m_2 - a}.$$

Eclu, if, oghaver, the parentts (11) for $j = 2$ rencants toluko for nekotopsix shavenue of C_1, to ypabnening (2) will umeet kas noluko menay hclomophic pewenind, a saburimity of the prourisolopic constount C_2, sas etm there are distinct realzomal kopnu etnenters to C_1, ypabnening (11) for $j = 2$.

Let now α — a real number such that $\Delta_{mn} = 0$ for (m_1, n_1), (m_2, n_2), $\dots$, (m_p, n_p), where $m_1 < m_2 < \dots < m_p$, $n_2 > n_2 > \dots > n_p$, $p \geqslant 1$. It is not difficult to see, that if $m_p \neq n_p$, then equation (2) has a holomorphic solution (3) if and only if

$$(m_k - a)\varphi_{m_kn_k} + b\overline{\varphi}_{n_km_k} = 0 \quad (k = 1, 2, \dots, p), \tag{12}$$

and the solution contains p arbitrary constoannts $C_1, C_2, \dots, C_p$, anmento $a_{m_kn_k} = C_k$ $(k = 1, 2, \dots, p)$. Let us emphasize that we did not take into account those pairs (m, n), which are obtained by permuting m and n.

Suppose that, that $m_p = n_p$ and the conditions (12) are fulfilled. To determine $a_{m_pn_p}$ we have the equation

$$(m_p - \overline{a})a_{m_pn_p} - b\overline{a}_{n_pm_p} = \varphi_{m_pn_p},$$

whence, as before, we obtain

$$a_{m_pn_p} = \left(C_p + \frac{|\varphi_{m_gn_p}|}{2}\, i\right)\frac{e^{\frac{i\varphi}{2}}}{\overline{b}},$$

where

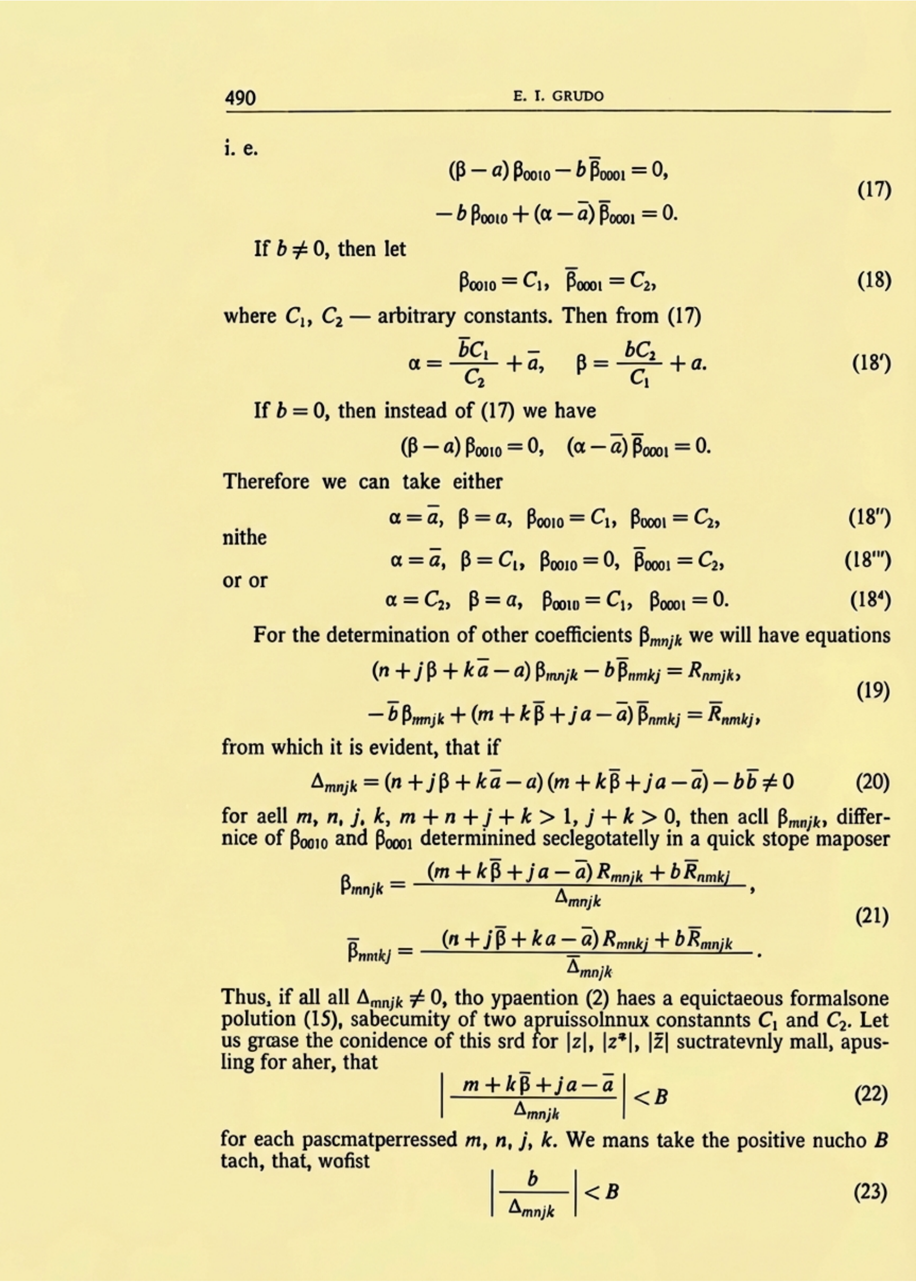
$$e^{i\varphi} = \frac{m_p - \overline{a}}{b} = \frac{\overline{b}}{m_p - a}$$

and C_p — is an arbitrary real constant.

Thus, if $m_p = n_p$, then conditions (12) are also necessary and sufficient for equation (2) to have a holomorphic solution (3), and this solution will depend on p arbitrary constounts $C_1, C_2, \dots, C_p$ and the constoannt C_p is realesome. Let now α — a will depend on p arbitrary constants

$$\Delta_{mn} = 0 \quad \text{for} \quad (m_1, n_1),\ (m_2, n_2),\ \dots,\ (m_p, n_p),\ \ m_1 < m_2 < \dots < m_p,$$
$$n_1 > n_2 > \dots > n_k,\ n_{k+1} > n_{k+2} > \dots > n_p,\ \ n_p > n_1,\ \ 1 \leqslant k < p.$$

Figure 4: Figure 4



i. e.

$$(\beta - a)\,\beta_{0010} - b\,\bar{\beta}_{0001} = 0, \qquad -b\,\beta_{0010} + (\alpha - \bar{a})\,\bar{\beta}_{0001} = 0. \tag{17}$$

If $b \neq 0$, then let

$$\beta_{0010} = C_1, \quad \bar{\beta}_{0001} = C_2, \tag{18}$$

where C_1, C_2 — arbitrary constants. Then from (17)

$$\alpha = \frac{\bar{b}C_1}{C_2} + \bar{a}, \qquad \beta = \frac{bC_2}{C_1} + a. \tag{18'}$$

If $b = 0$, then instead of (17) we have

$$(\beta - a)\,\beta_{0010} = 0, \quad (\alpha - \bar{a})\,\bar{\beta}_{0001} = 0.$$

Therefore we can take either

$$\alpha = \bar{a}, \ \beta = a, \ \beta_{0010} = C_1, \ \beta_{0001} = C_2, \tag{18''}$$

nithe

$$\alpha = \bar{a}, \ \beta = C_1, \ \beta_{0010} = 0, \ \bar{\beta}_{0001} = C_2, \tag{18'''}$$

or or

$$\alpha = C_2, \ \beta = a, \ \beta_{0010} = C_1, \ \beta_{0001} = 0. \tag{18⁴}$$

For the determination of other coefficients β_{mnjk} we will have equations

$$(n + j\beta + k\bar{a} - a)\,\beta_{mnjk} - b\,\bar{\beta}_{nmkj} = R_{nmjk}, \qquad -\bar{b}\,\beta_{mnjk} + (m + k\bar{\beta} + ja - \bar{a})\,\bar{\beta}_{nmkj} = \bar{R}_{nmkj}, \tag{19}$$

from which it is evident, that if

$$\Delta_{mnjk} = (n + j\beta + k\bar{a} - a)(m + k\bar{\beta} + ja - \bar{a}) - b\bar{b} \neq 0 \tag{20}$$

for aell m, n, j, k, $m + n + j + k > 1$, $j + k > 0$, then acll β_{mnjk}, differnice of β_{0010} and β_{0001} determinined seclegotatelly in a quick stope maposer

$$\beta_{mnjk} = \frac{(m + k\bar{\beta} + ja - \bar{a})\,R_{mnjk} + b\,\bar{R}_{nmkj}}{\Delta_{mnjk}}, \qquad \bar{\beta}_{nmkj} = \frac{(n + j\bar{\beta} + ka - \bar{a})\,R_{mnkj} + b\,\bar{R}_{mnjk}}{\bar{\Delta}_{mnjk}}. \tag{21}$$

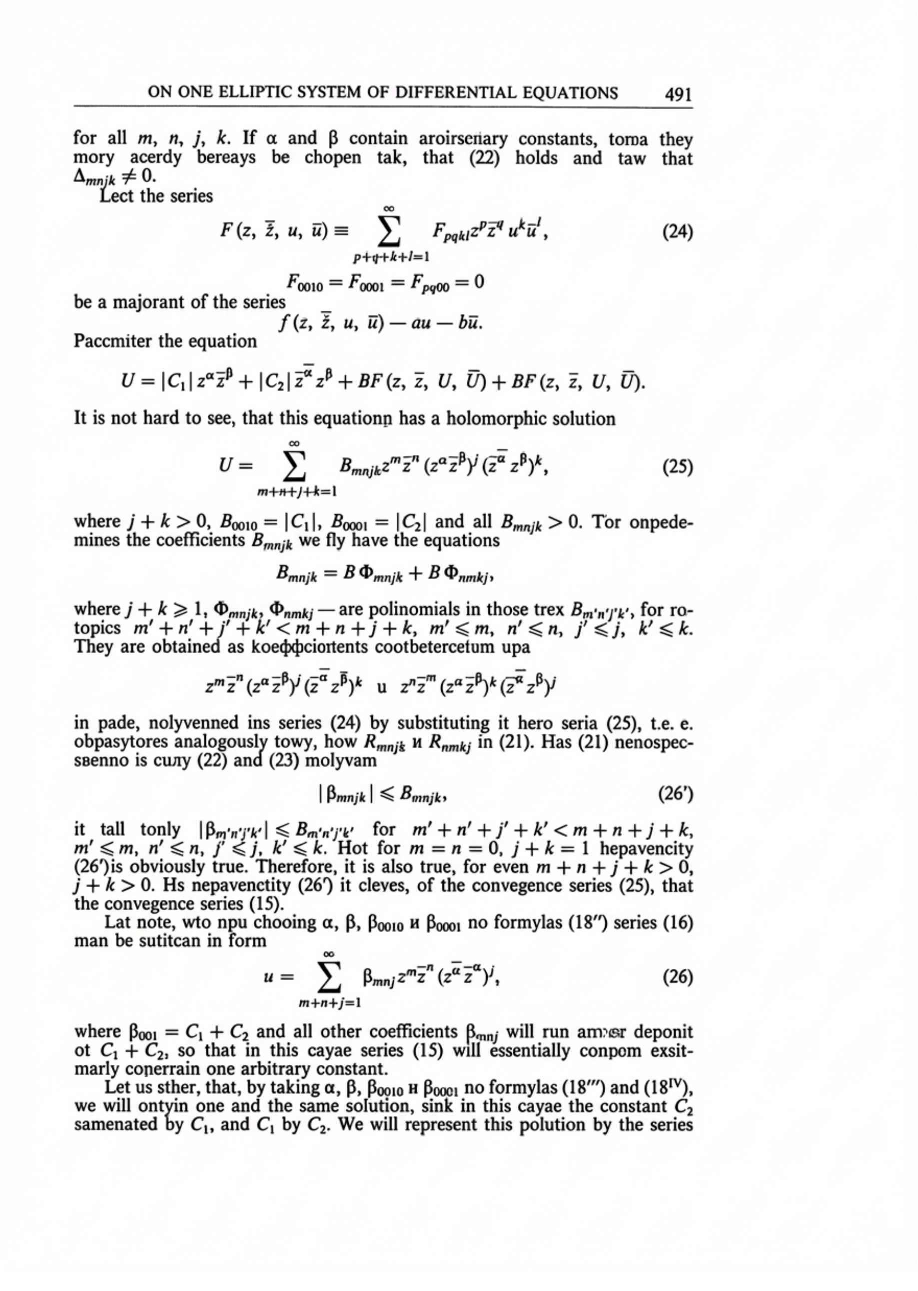
Thus, if all all $\Delta_{mnjk} \neq 0$, tho ypaention (2) haes a equictaeous formalsone polution (15), sabecumity of two apruissolnnux constannts C_1 and C_2. Let us grœse the conidence of this srd for $|z|$, $|z^*|$, $|\bar{z}|$ suctratevnly mall, apusling for aher, that

$$\left| \frac{m + k\bar{\beta} + ja - \bar{a}}{\Delta_{mnjk}} \right| < B \tag{22}$$

for each pascmatperressed m, n, j, k. We mans take the positive nucho B tach, that, wofist

$$\left| \frac{b}{\Delta_{mnjk}} \right| < B \tag{23}$$

Figure 5: Figure 5



for all m, n, j, k. If α and β contain aroirseriary constants, toma they mory acerdy bereays be chopen tak, that (22) holds and taw that $\Delta_{mnjk} \neq 0$.

Lect the series

$$F(z, \bar{\bar{z}}, u, \bar{u}) \equiv \sum_{p+q+k+l=1}^{\infty} F_{pqkl} z^p \bar{z}^q u^k \bar{u}^l, \tag{24}$$

$$F_{0010} = F_{0001} = F_{pq00} = 0$$

be a majorant of the series

$$f(z, \bar{\bar{z}}, u, \bar{u}) - au - b\bar{u}.$$

Paccmiter the equation

$$U = |C_1| z^{\alpha} \bar{z}^{\beta} + |C_2| \bar{z}^{\bar{\alpha}} z^{\beta} + BF(z, \bar{z}, U, \bar{U}) + BF(z, \bar{z}, U, \bar{U}).$$

It is not hard to see, that this equationn has a holomorphic solution

$$U = \sum_{m+n+j+k=1}^{\infty} B_{mnjk} z^m \bar{z}^n (z^{\alpha} \bar{z}^{\beta})^j (\bar{z}^{\bar{\alpha}} z^{\beta})^k, \tag{25}$$

where $j + k > 0$, $B_{0010} = |C_1|$, $B_{0001} = |C_2|$ and all $B_{mnjk} > 0$. Tor onpedemines the coefficients B_{mnjk} we fly have the equations

$$B_{mnjk} = B\,\Phi_{mnjk} + B\,\Phi_{nmkj},$$

where $j + k \geqslant 1$, Φ_{mnjk}, Φ_{nmkj} — are polinomials in those trex $B_{m'n'j'k'}$, for rotopics $m' + n' + j' + k' < m + n + j + k$, $m' \leqslant m$, $n' \leqslant n$, $j' \leqslant j$, $k' \leqslant k$. They are obtained as koeффciortents cootbetercetum upa

$$z^m \bar{z}^n (z^{\alpha} \bar{z}^{\beta})^j (\bar{z}^{\bar{\alpha}} z^{\bar{\beta}})^k \quad \text{u} \quad z^n \bar{z}^m (z^{\alpha} \bar{z}^{\beta})^k (\bar{z}^{\bar{\alpha}} z^{\beta})^j$$

in pade, nolyvenned ins series (24) by substituting it hero seria (25), t.e. e. obpasytores analogously towy, how R_{mnjk} и R_{nmkj} in (21). Has (21) nenospecsвenno is cuлy (22) and (23) molyvam

$$|\beta_{mnjk}| \leqslant B_{mnjk}, \tag{26'}$$

it tall tonly $|\beta_{m'n'j'k'}| \leqslant B_{m'n'j'k'}$ for $m' + n' + j' + k' < m + n + j + k$, $m' \leqslant m$, $n' \leqslant n$, $j' \leqslant j$, $k' \leqslant k$. Hot for $m = n = 0$, $j + k = 1$ hepavencity (26′)is obviously true. Therefore, it is also true, for even $m + n + j + k > 0$, $j + k > 0$. Hs nepavenctity (26′) it cleves, of the convegence series (25), that the convegence series (15).

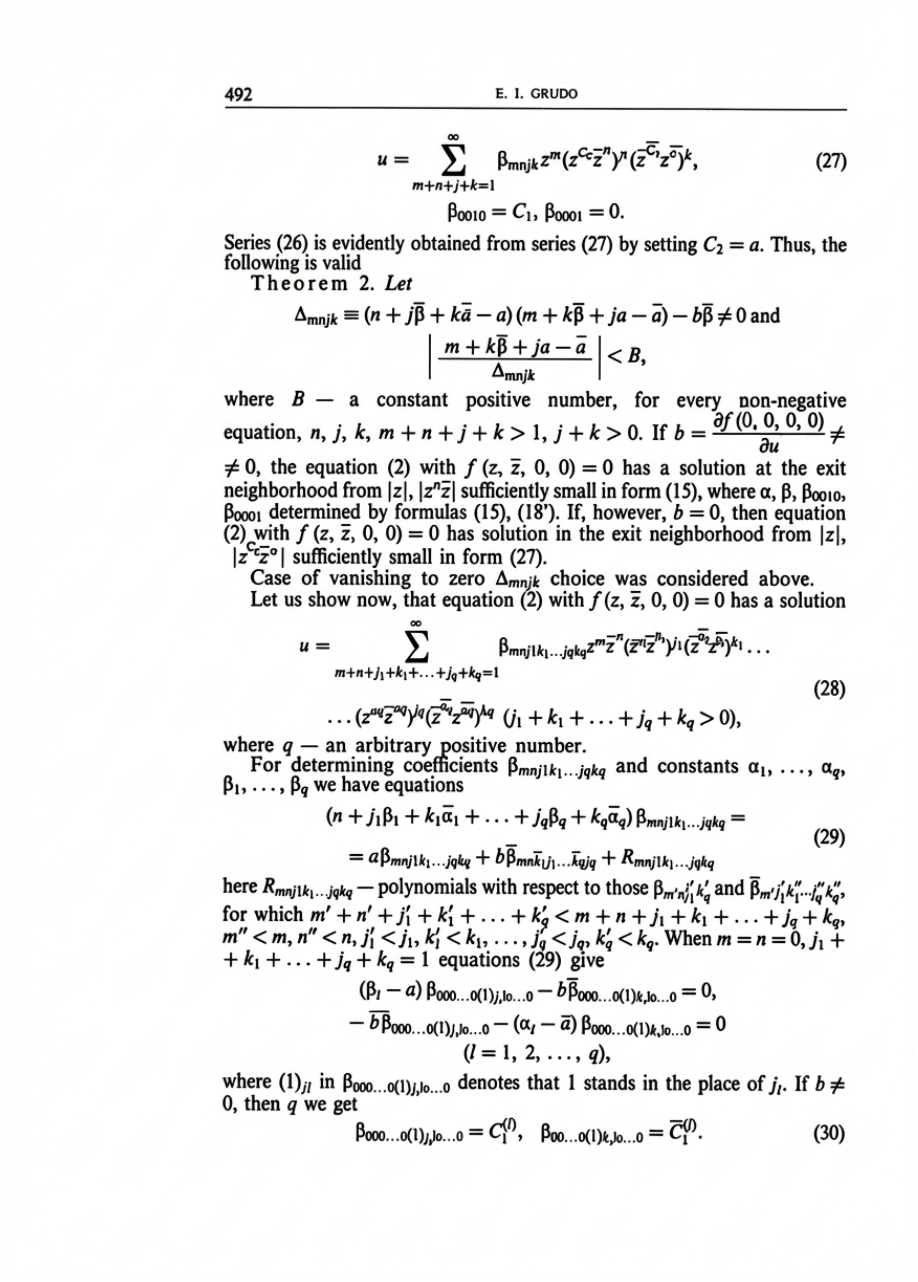
Lat note, wto npu chooing α, β, β_{0010} и β_{0001} no formylas (18″) series (16) man be sutitcan in form

$$u = \sum_{m+n+j=1}^{\infty} \beta_{mnj} z^m \bar{z}^n (z^{\bar{\alpha}} \bar{z}^{-\alpha})^j, \tag{26}$$

where $\beta_{001} = C_1 + C_2$ and all other coefficients β_{mnj} will run amser deponit ot $C_1 + C_2$, so that in this cayae series (15) will essentially conpom exsitmarly conerrain one arbitrary constant.

Let us sther, that, by taking α, β, β_{0010} н β_{0001} no formylas (18‴) and (18^{IV}), we will ontyin one and the same solution, sink in this cayae the constant C_2 samenated by C_1, and C_1 by C_2. We will represent this polution by the series

Figure 6: Figure 6



$$u = \sum_{m+n+j+k=1}^{\infty} \beta_{mnjk} z^m (z^{C_c}\bar{z}^{-n})^n (\bar{z}^{\bar{C}_1} z^{\bar{o}})^k, \tag{27}$$

$$\beta_{0010} = C_1,\ \beta_{0001} = 0.$$

Series (26) is evidently obtained from series (27) by setting $C_2 = a$. Thus, the following is valid

Theorem 2. *Let*

$$\Delta_{mnjk} \equiv (n + j\bar{\beta} + k\bar{a} - a)(m + k\bar{\beta} + ja - \bar{a}) - b\bar{\beta} \neq 0 \text{ and}$$

$$\left| \frac{m + k\bar{\beta} + ja - \bar{a}}{\Delta_{mnjk}} \right| < B,$$

where B — a constant positive number, for every non-negative equation, n, j, k, $m + n + j + k > 1$, $j + k > 0$. If $b = \dfrac{\partial f(0, 0, 0, 0)}{\partial u} \neq$ $\neq 0$, the equation (2) with $f(z, \bar{z}, 0, 0) = 0$ has a solution at the exit neighborhood from $|z|$, $|z^n\bar{z}|$ sufficiently small in form (15), where α, β, β_{0010}, β_{0001} determined by formulas (15), (18'). If, however, $b = 0$, then equation (2) with $f(z, \bar{z}, 0, 0) = 0$ has solution in the exit neighborhood from $|z|$, $|z^{C_c}\bar{z}^{o}|$ sufficiently small in form (27).

Case of vanishing to zero Δ_{mnjk} choice was considered above.

Let us show now, that equation (2) with $f(z, \bar{z}, 0, 0) = 0$ has a solution

$$u = \sum_{m+n+j_1+k_1+\ldots+j_q+k_q=1}^{\infty} \beta_{mnj1k_1\ldots jqkq} z^m \bar{z}^{-n} (\bar{z}^{n}\bar{z}^{-n_1})^{j_1} (\bar{z}^{-\bar{a}_2}\bar{z}^{\bar{a}_1})^{k_1} \ldots$$
$$\ldots (z^{aq}\bar{z}^{-aq})^{jq} (\bar{z}^{-\bar{a}_q} z^{\bar{a}q})^{kq}\ (j_1 + k_1 + \ldots + j_q + k_q > 0), \tag{28}$$

where q — an arbitrary positive number.

For determining coefficients $\beta_{mnj1k_1\ldots jqkq}$ and constants $\alpha_1, \ldots, \alpha_q$, $\beta_1, \ldots, \beta_q$ we have equations

$$(n + j_1\beta_1 + k_1\bar{\alpha}_1 + \ldots + j_q\beta_q + k_q\bar{\alpha}_q)\,\beta_{mnj1k_1\ldots jqkq} =$$
$$= a\beta_{mnj1k_1\ldots jqkq} + b\bar{\beta}_{mn\bar{k}_1j_1\ldots\bar{k}qjq} + R_{mnj1k_1\ldots jqkq} \tag{29}$$

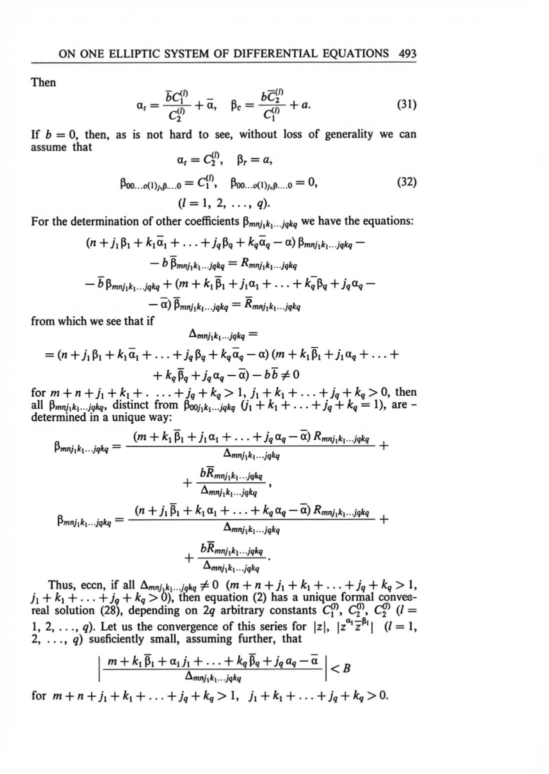
here $R_{mnj1k_1\ldots jqkq}$ — polynomials with respect to those $\beta_{m'n'j_1'k_q'}$ and $\bar{\beta}_{m'j_1'k_1''\ldots j_q''k_q''}$, for which $m' + n' + j_1' + k_1' + \ldots + k_q' < m + n + j_1 + k_1 + \ldots + j_q + k_q$, $m'' < m$, $n'' < n$, $j_1' < j_1$, $k_l' < k_1, \ldots, j_q' < j_q$, $k_q' < k_q$. When $m = n = 0$, $j_1 + k_1 + \ldots + j_q + k_q = 1$ equations (29) give

$$(\beta_l - a)\,\beta_{000\ldots0(1)_{jl}0\ldots0} - b\bar{\beta}_{000\ldots0(1)_{kl}0\ldots0} = 0,$$
$$-\overline{b}\bar{\beta}_{000\ldots0(1)_{jl}0\ldots0} - (\alpha_l - \bar{a})\,\beta_{000\ldots0(1)_{kl}0\ldots0} = 0$$
$$(l = 1, 2, \ldots, q),$$

where $(1)_{jl}$ in $\beta_{000\ldots0(1)_{jl}0\ldots0}$ denotes that 1 stands in the place of j_l. If $b \neq 0$, then q we get

$$\beta_{000\ldots0(1)_{jl}0\ldots0} = C_1^{(l)},\quad \beta_{00\ldots0(1)_{kl}0\ldots0} = \bar{C}_1^{(l)}. \tag{30}$$

Figure 7: Figure 7



Then

$$\alpha_t = \frac{\bar{b}C_1^{(l)}}{C_2^{(l)}} + \bar{\alpha}, \quad \beta_c = \frac{b\bar{C}_2^{(l)}}{C_1^{(l)}} + a. \tag{31}$$

If $b = 0$, then, as is not hard to see, without loss of generality we can assume that

$$\alpha_t = C_2^{(l)}, \quad \beta_r = a,$$
$$\beta_{00\ldots o(1)_{j_l}\beta\ldots 0} = C_1^{(l)}, \quad \beta_{00\ldots o(1)_{j_o}\beta\ldots 0} = 0, \tag{32}$$
$$(l = 1,\ 2,\ \ldots,\ q).$$

For the determination of other coefficients $\beta_{mnj_1k_1\ldots jqkq}$ we have the equations:

$$(n + j_1\beta_1 + k_1\bar{\alpha}_1 + \ldots + j_q\beta_q + k_q\bar{\alpha}_q - \alpha)\,\beta_{mnj_1k_1\ldots jqkq} -$$
$$- b\,\bar{\beta}_{mnj_1k_1\ldots jqkq} = R_{mnj_1k_1\ldots jqkq}$$
$$- \bar{b}\,\beta_{mnj_1k_1\ldots jqkq} + (m + k_1\bar{\beta}_1 + j_1\alpha_1 + \ldots + k_q\bar{\beta}_q + j_q\alpha_q -$$
$$- \bar{\alpha})\,\bar{\beta}_{mnj_1k_1\ldots jqkq} = \bar{R}_{mnj_1k_1\ldots jqkq}$$

from which we see that if

$$\Delta_{mnj_1k_1\ldots jqkq} =$$
$$= (n + j_1\beta_1 + k_1\bar{\alpha}_1 + \ldots + j_q\beta_q + k_q\bar{\alpha}_q - \alpha)\,(m + k_1\bar{\beta}_1 + j_1\alpha_q + \ldots +$$
$$+ k_q\bar{\beta}_q + j_q\alpha_q - \bar{\alpha}) - b\bar{b} \neq 0$$

for $m + n + j_1 + k_1 + .\ \ldots + j_q + k_q > 1$, $j_1 + k_1 + \ldots + j_q + k_q > 0$, then all $\beta_{mnj_1k_1\ldots jqkq}$, distinct from $\beta_{00j_1k_1\ldots jqkq}$ $(j_1 + k_1 + \ldots + j_q + k_q = 1)$, are - determined in a unique way:

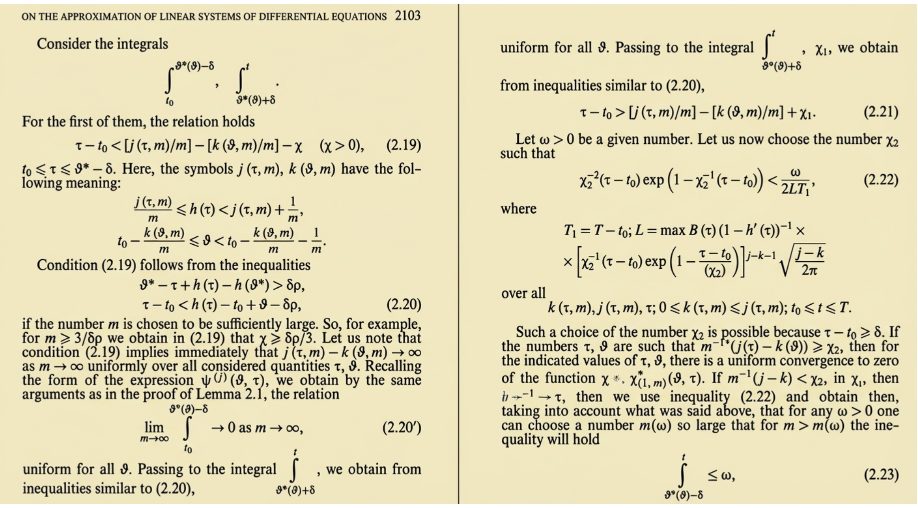
$$\beta_{mnj_1k_1\ldots jqkq} = \frac{(m + k_1\bar{\beta}_1 + j_1\alpha_1 + \ldots + j_q\alpha_q - \bar{\alpha})\,R_{mnj_1k_1\ldots jqkq}}{\Delta_{mnj_1k_1\ldots jqkq}} +$$
$$+ \frac{b\bar{R}_{mnj_1k_1\ldots jqkq}}{\Delta_{mnj_1k_1\ldots jqkq}},$$
$$\beta_{mnj_1k_1\ldots jqkq} = \frac{(n + j_1\bar{\beta}_1 + k_1\alpha_1 + \ldots + k_q\alpha_q - \bar{\alpha})\,R_{mnj_1k_1\ldots jqkq}}{\Delta_{mnj_1k_1\ldots jqkq}} +$$
$$+ \frac{b\bar{R}_{mnj_1k_1\ldots jqkq}}{\Delta_{mnj_1k_1\ldots jqkq}}.$$

Thus, eccn, if all $\Delta_{mnj_1k_1\ldots jqkq} \neq 0$ $(m + n + j_1 + k_1 + \ldots + j_q + k_q > 1$, $j_1 + k_1 + \ldots + j_q + k_q > 0)$, then equation (2) has a unique formal conveв-real solution (28), depending on $2q$ arbitrary constants $C_1^{(l)}$, $C_2^{(l)}$, $C_2^{(l)}$ $(l = 1, 2, \ldots, q)$. Let us the convergence of this series for $|z|$, $|z^{\alpha_l}\bar{z}^{\beta_l}|$ $(l = 1, 2, \ldots, q)$ susficiently small, assuming further, that

$$\left|\frac{m + k_1\bar{\beta}_1 + \alpha_1 j_1 + \ldots + k_q\bar{\beta}_q + j_q a_q - \bar{\alpha}}{\Delta_{mnj_1k_1\ldots jqkq}}\right| < B$$

for $m + n + j_1 + k_1 + \ldots + j_q + k_q > 1$, $j_1 + k_1 + \ldots + j_q + k_q > 0$.

Figure 8: Figure 8



Consider the integrals

$$\int_{t_0}^{\vartheta^*(\vartheta)-\delta},\quad \int_{\vartheta^*(\vartheta)+\delta}^{t}.$$

For the first of them, the relation holds

$$\tau - t_0 < [j(\tau, m)/m] - [k(\vartheta, m)/m] - \chi \quad (\chi > 0), \tag{2.19}$$

$t_0 \leqslant \tau \leqslant \vartheta^* - \delta$. Here, the symbols $j(\tau, m)$, $k(\vartheta, m)$ have the following meaning:

$$\frac{j(\tau, m)}{m} \leqslant h(\tau) < j(\tau, m) + \frac{1}{m},$$
$$t_0 - \frac{k(\vartheta, m)}{m} \leqslant \vartheta < t_0 - \frac{k(\vartheta, m)}{m} - \frac{1}{m}.$$

Condition (2.19) follows from the inequalities

$$\vartheta^* - \tau + h(\tau) - h(\vartheta^*) > \delta\rho,$$
$$\tau - t_0 < h(\tau) - t_0 + \vartheta - \delta\rho, \tag{2.20}$$

if the number m is chosen to be sufficiently large. So, for example, for $m \geqslant 3/\delta\rho$ we obtain in (2.19) that $\chi \geqslant \delta\rho/3$. Let us note that condition (2.19) implies immediately that $j(\tau, m) - k(\vartheta, m) \to \infty$ as $m \to \infty$ uniformly over all considered quantities τ, ϑ. Recalling the form of the expression $\psi^{(j)}(\vartheta, \tau)$, we obtain by the same arguments as in the proof of Lemma 2.1, the relation

$$\lim_{m\to\infty} \int_{t_0}^{\vartheta^*(\vartheta)-\delta} \to 0 \text{ as } m \to \infty, \tag{2.20'}$$

uniform for all ϑ. Passing to the integral $\int_{\vartheta^*(\vartheta)+\delta}^{t}$, we obtain from inequalities similar to (2.20),

uniform for all ϑ. Passing to the integral $\int_{\vartheta^*(\vartheta)+\delta}^{t}$, χ_1, we obtain from inequalities similar to (2.20),

$$\tau - t_0 > [j(\tau, m)/m] - [k(\vartheta, m)/m] + \chi_1. \tag{2.21}$$

Let $\omega > 0$ be a given number. Let us now choose the number χ_2 such that

$$\chi_2^{-2}(\tau - t_0)\exp\left(1 - \chi_2^{-1}(\tau - t_0)\right) < \frac{\omega}{2LT_1}, \tag{2.22}$$

where

$$T_1 = T - t_0;\ L = \max B(\tau)\,(1 - h'(\tau))^{-1} \times$$
$$\times \left[\chi_2^{-1}(\tau - t_0)\exp\left(1 - \frac{\tau - t_0}{(\chi_2)}\right)\right]^{j-k-1}\sqrt{\frac{j-k}{2\pi}}$$

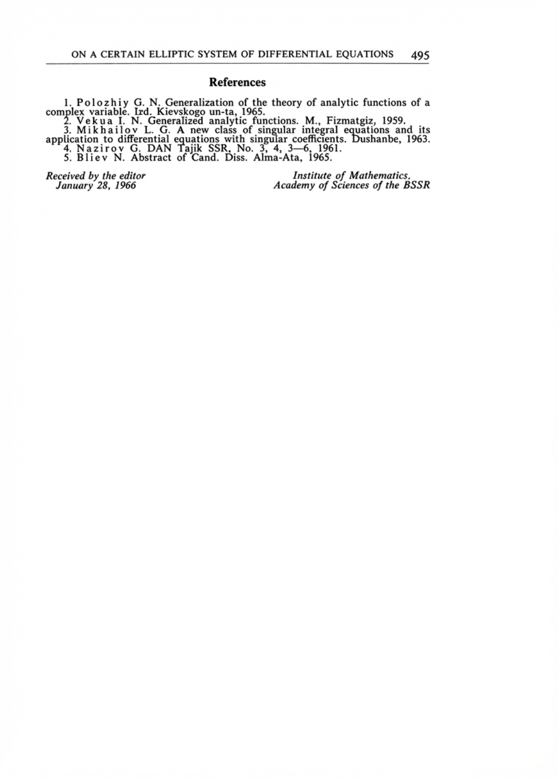
over all

$$k(\tau, m), j(\tau, m), \tau;\ 0 \leqslant k(\tau, m) \leqslant j(\tau, m);\ t_0 \leqslant t \leqslant T.$$

Such a choice of the number χ_2 is possible because $\tau - t_0 \geqslant \delta$. If the numbers τ, ϑ are such that $m^{-1*}(j(\tau) - k(\vartheta)) \geqslant \chi_2$, then for the indicated values of τ, ϑ, there is a uniform convergence to zero of the function χ *. $\chi^*_{(1,m)}(\vartheta, \tau)$. If $m^{-1}(j - k) < \chi_2$, in χ_1, then $b \to^{-1} \to \tau$, then we use inequality (2.22) and obtain then, taking into account what was said above, that for any $\omega > 0$ one can choose a number $m(\omega)$ so large that for $m > m(\omega)$ the inequality will hold

$$\int_{\vartheta^*(\vartheta)-\delta}^{t} \leq \omega, \tag{2.23}$$

Figure 9: Figure 9



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Received by the editor
January 28, 1966

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Figure 10: Figure 10