ON A REALIZATION BY ENTIRE FUNCTIONS OF INFINITE-DIMENSIONAL REPRESENTATIONS OF THE ROTATION GROUP WITH COMPLEX SPIN

MATHEMATICAL PHYSICS

1967

SovietRxiv

Abstract

Full Text

UDC 519.46

MATHEMATICAL PHYSICS

S. S. SANNIKOV

ON A REALIZATION BY ENTIRE FUNCTIONS OF INFINITE-DIMENSIONAL REPRESENTATIONS OF THE ROTATION GROUP WITH COMPLEX SPIN

(Presented by Academician N. N. Bogolyubov, 24 XII 1966)

In ([1]) the author constructed new representations $D(\lambda)$ of the rotation group R_3, corresponding to arbitrary (in the general case complex) values of the weight λ.* The realization considered in ([1]) has the drawback that the class of functions in which the representation $D(\lambda)$ is defined is formed by many-valued analytic functions of a complex variable with singular points of the type of branch points, situated in any part of the complex plane. The uniformization which must then be carried out makes it difficult to analyze the properties of the representations.

In the special case of representations with $\lambda = -{}^1/_4$ and $\lambda = -{}^3/_4$, in ([3],[4]) a realization was constructed by single-valued entire analytic functions of a complex variable (to it leads the operation of extracting the square root from a spinor). It turns out that this realization can be extended to representations with arbitrary weight. A realization by entire functions is of considerable interest (in this realization, for example, there are no well-known finite-dimensional representations ([5])).

Here we shall confine ourselves to deriving the basic formulas of this realization. They will be useful in constructing a consistent theory of representations of the rotation group with complex spin.

Let $u \to T_\lambda(u)$, where

$$u = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1,$$

be a representation of the group R_3** of weight λ in the realization under consideration. The starting formula is the definition of the transformation $T_\lambda(u)$ on the function $f(z) = 1$ $(z = x + iy)$:

$$T_\lambda(u) \cdot 1 = \varkappa_\lambda(z; u) = \bar{\alpha}^{2\lambda} \exp\left(\frac{1}{2} z^2 \frac{\beta}{\alpha}\right). \qquad (1)$$

For $\lambda = -{}^1/_4$, formula (1) goes over into the corresponding formula of ([3]). The expansion of expression (1) in a series in powers of z can be written in the form***

$$\varkappa_\lambda(z;u) = \sum_{n=0}^{\infty} D_{n0}^{(\lambda)}(u) f_n^{(\lambda)}(z), \tag{2}$$

* Representations with complex spin (or complex angular momentum) are used in the description of unstable quantum-mechanical systems ([2]) (finite-dimensional representations are used in the description of stable systems).

** Strictly speaking, $T_\lambda(u)$ is a representation of the group $SU(2)$. In view of the local isomorphism of the groups R_3 and $SU(2)$, we shall speak of representations of the group $SU(2)$ as representations of the group R_3.

*** For rational $\lambda = l/s$, the functions (1), (3) are s-valued on the group R_3; for irrational and complex λ these functions are infinitely many-valued ([1]). It is known that the groups Lie groups are analytic multiforms (Hilbert' s fifth problem). Therefore, for them we can introduce the concept of a Riemann surface. On the corresponding to a given λ Riemann surface, the functions (1), (3) are single-valued.

where $D_{nm}^{(\lambda)}(u)$ are the matrix elements of the representation of weight λ ([1]), in particular

$$D_{n0}^{(\lambda)}(u) = (-i)^n \sqrt{\frac{\Gamma(n-2\lambda)}{n!\Gamma(-2\lambda)}}\, \bar{\alpha}^{2\lambda-n} \beta^n, \tag{3}$$

and $f_n^{(\lambda)}(z)$ are the functions of the canonical basis of our new realization. From (2) we find an explicit expression for $f_n^{(\lambda)}(z)$:

$$f_n^{(\lambda)}(z) = \frac{i^n}{2^n} \sqrt{\frac{\Gamma(-2\lambda)}{n!\Gamma(n-2\lambda)}}\, z^{2n}, \qquad n = 0, 1, 2, \dots . \tag{4}$$

Let us now construct the operator mapping the old realization ([1]) into the new one. It has the form ($\zeta = \xi + i\eta$)

$$K_\lambda(z;\zeta) = \sum_{n,m=0}^{\infty} g_{nm}^{(\lambda)} f_n^{(\lambda)}(z) \overline{f_m^{(\lambda)}(\zeta)} = \exp\left(-\frac{1}{2} z^2 \bar{\zeta}\right), \tag{5}$$

where

$$f_m^{(\lambda)}(\zeta) = i^m \sqrt{\frac{\Gamma(m-2\lambda)}{m!\Gamma(-2\lambda)}}\, \zeta^m$$

is the canonical basis of the realization ([1]), and

$$g_{mn}^{(\lambda)} = (-1)^n \eta_n^{(\lambda)} \delta_{mn}$$

is the fundamental metric form of the representation of weight λ ([1]) ($\eta_n^{(\lambda)} = (-1)^n$ if $n < 2[\operatorname{Re}\lambda]$, and $\eta_n^{(\lambda)} = 1$ if $n \geq 2[\operatorname{Re}\lambda]+1$). The operator (5) has the property that every analytic function $f^{(\lambda)}(\zeta)$, defined in its Mittag-Leffler star of the variable ζ, is transformed by it into an entire analytic function $f^{(\lambda)}(z)$, defined in the whole complex z-plane:

$$\int K_\lambda(z;\zeta) f^{(\lambda)}(\zeta)\, d\mu_\lambda(\zeta) = f^{(\lambda)}(z), \tag{6}$$

where

$$d\mu_\lambda(\zeta) = \frac{1}{\pi}(1+|\zeta|^2)^{-2(\lambda+1)}\, d\xi, \qquad d\zeta = d\xi\, d\eta$$

is the invariant measure on ζ ([1]).

Let us now find the invariant measure in the new realization. The functions of the canonical basis (4) must be normalized by the condition ([1])

$$[f_m^{(\lambda)}, f_n^{(\lambda)}]_\lambda = \int \overline{(I f_m^{(\lambda)}(z))}\, f_n^{(\lambda)}(z)\, d\mu_\lambda(z) = g_{mn}^{(\lambda)}, \tag{7}$$

where $If(z) = f(iz)$, and $d\mu_\lambda(z) = \rho_\lambda(z)\, dz$, $dz = dx\, dy$, is the sought measure. From (7) there follows the relation

$$\int_0^\infty v^{2n} \rho_\lambda(v)\, dv = \frac{2^{2n}\Gamma(n+1)\Gamma(n-2\lambda)}{\pi\Gamma(-2\lambda)}, \qquad v = |z|^2. \tag{8}$$

Continuing (8) analytically into the right half-plane of the complex variable n and applying the Mellin inversion formula to (8), we obtain ([6]) ($a > 2\operatorname{Re}\lambda$)

$$\begin{aligned} \rho_\lambda(v) &= \frac{1}{i\pi^2\Gamma(-2\lambda)} \int_{a-i\infty}^{a+i\infty} 2^{2\nu}\Gamma(\nu+1)\Gamma(\nu-2\lambda) v^{-2\nu-1}\, d\nu \\ &= \frac{2}{\pi\Gamma(-2\lambda)} \left(\frac{2}{v}\right)^{2\lambda} K_{2\lambda+1}(v), \end{aligned} \tag{9}$$

where $K_{2\lambda+1}(v)$ are Macdonald functions.*

The operator inverse to the operator $K_\lambda(z;\zeta)$ will be

$$K_\lambda^{-1}(\zeta; z) = \exp\left(\frac{1}{2} z^2 \bar{\zeta}\right), \tag{10}$$

and in this case we have

$$\int K_\lambda(z;\zeta) K_\lambda^{-1}(\zeta; z')\, d\mu_\lambda(\zeta) = \Gamma(-2\lambda) \left(\frac{z\bar{z}'}{2}\right)^{2\lambda+1} I_{-2\lambda-1}(z\bar{z}') = \delta_\lambda(z, z'), \tag{11}$$

* Obviously, in formulas (7), (8) one must have $n > 2\operatorname{Re}\lambda$ ($n \geq 0$). If $n < 2\operatorname{Re}\lambda$, then the right-hand sides in formulas (7), (8) are equal to the regularized values of the integrals (for the definition of a regularized integral, see ([7])).

where $I_{-2\lambda-1}$ is the Bessel function of imaginary argument. The functional (11) has the property of a δ-functional, i.e.,

$$\int \delta_\lambda(z, z') f^{(\lambda)}(z')\, d\mu_\lambda(z') = f^{(\lambda)}(z)$$

($f^{(\lambda)}(z)$ is an entire analytic function).

We now define the infinitesimal operators of the representation in the new realization. By definition we have

$$L_\alpha^{(\lambda)}(z) = \int K_\lambda(z,\xi) L_\alpha^{(\lambda)}(\xi) K_\lambda^{-1}(\xi; z')\, d\mu_\lambda(\xi), \qquad \alpha = 1, 2, 3, \tag{12}$$

where $L^{(\lambda)\alpha}(\xi)$ are the infinitesimal operators of the former realization (1) ($L_\pm = L_1 \pm iL_2$):

$$L_3^{(\lambda)}(\xi) = \xi\, d/d\xi - \lambda, \quad L_+^{(\lambda)}(\xi) = \xi^2\, d/d\xi - 2\lambda\xi, \quad L_-^{(\lambda)}(\xi) = -d/d\xi. \tag{13}$$

Carrying out the integration with respect to ξ in (12), we find

$$L_3^{(\lambda)}(z) = \left(\frac{1}{2} z \frac{d}{dz} - \lambda\right) \delta_\lambda(z, z'), \qquad L_+^{(\lambda)}(z) = \frac{z^2}{2}\, \delta_\lambda(z, z'),$$

$$L_-^{(\lambda)}(z) = \delta_\lambda(z, z')(-{z'}^2/2). \tag{14}$$

The last expression can also be written in the form

$$L_-^{(\lambda)}(z) = -\frac{1}{2} \left(\frac{d^2}{dz^2} - (4\lambda + 1)\frac{1}{z}\frac{d}{dz}\right) \delta_\lambda(z, z'). \tag{14'}$$

The operators (14) satisfy the well-known commutation relations $[L_\alpha, L_\beta] = i\varepsilon_{\alpha\beta\gamma} L_\gamma$; in addition, $L^2 = L_1^2 + L_2^2 + L_3^2 = \lambda(\lambda + 1)$. On the functions of the canonical basis (4), the operators L_α act according to the formulas

$$L_3 f_n^{(\lambda)} = (n - \lambda) f_n^{(\lambda)}, \qquad L_+ f_n^{(\lambda)} = a_{n+1}^{(\lambda)} f_{n+1}^{(\lambda)}, \qquad L_- f_n^{(\lambda)} = a_n^{(\lambda)} f_{n-1}^{(\lambda)}, \tag{15}$$

where

$$a_n^{(\lambda)} = -i\sqrt{n(n-2\lambda-1)}.$$

For $\lambda = -\frac{1}{4}$ we introduce two operators a_λ^1, a_λ^2, satisfying the commutation relation

$$[a_\lambda^1, a_\lambda^2] = 1 \tag{16}$$

and possessing the property that, as in (3),

$$L_\alpha^{(\lambda)}(z) = \tfrac{1}{4}(\sigma_\alpha)_i^k a_\lambda^i a_\lambda^l \varepsilon_{kl}, \tag{17}$$

where σ_α are the three Pauli matrices, and

$$\varepsilon_{kl} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Moreover,

$$[L_\alpha^{(\lambda)}(z), a_\lambda^i] = \tfrac{1}{2}(\sigma_\alpha)_k^i a_\lambda^k, \qquad \lambda = -\tfrac{1}{4}. \tag{18}$$

From (17), (18) we find*

$$a_\lambda^1 = z, \qquad a_\lambda^2 = -d/dz. \tag{19}$$

Under a transformation from the rotation group the operators a_λ^i are transformed as

$$T_\lambda(u) a_\lambda^i T_\lambda^{-1}(u) = u_k^i a_\lambda^k, \qquad \lambda = -\tfrac{1}{4}, \tag{20}$$

in particular, $z \to z' = \alpha z + \bar{\beta}\, d/dz$.

---

* Thus, if $K(z, d/dz)$ is the field of coordinate and momentum operators (with generators z and d/dz), then from our construction (formulas (17), (18)) it follows that $L_\alpha^{(\lambda)} \subset K(z, d/dz)$, i.e. the Lie algebra of the group $L_\alpha^{(\lambda)}$ is contained in the field K.

Now we can determine the result of the action of the operator $T_\lambda(u)$ on an arbitrary function $f^{(\lambda)}(z)$ of the representation $D(\lambda)$ in our realization by the formula

$$T_\lambda(u) f^{(\lambda)}(z) = T_\lambda(u) f^{(\lambda)}(z) T_\lambda^{-1}(u)(T_\lambda(u)\cdot 1) = f^{(\lambda)}(z') \varkappa_\lambda(z, u), \tag{21}$$

$${z'}^2 = d^2 z^2 + \beta^2 \left(\frac{d^2}{dz^2} - \frac{4\lambda+1}{z}\frac{d}{dz}\right) + 2\alpha\beta\left(z\frac{d}{dz} - 2\lambda\right).$$

In particular, on the functions of the canonical basis (4) the operator $T_\lambda(u)$ acts according to the formula

$$T_\lambda(u) f_n^{(\lambda)}(z) = p_n^{(\lambda)}(z^2; u)\, \varkappa_\lambda(z; u), \tag{22}$$

where $p_n^{(\lambda)}$ are polynomials of degree n in z^2. Calculations give for $p_n^{(\lambda)}$

$$p_n^{(\lambda)}(z^2; u) = \sqrt{\frac{\Gamma(n-2\lambda)}{n!\,\Gamma(-2\lambda)}} \left(i\frac{\bar{\beta}}{\alpha}\right)^n {}_1F_1\left(-n;\, -2\lambda;\, -\frac{z^2}{2\alpha\bar{\beta}}\right), \tag{23}$$

where ${}_1F_1(a; b; x)$ is the confluent hypergeometric function.

It follows from (22) that in the realization under consideration the representation $D(\lambda)$ is given in the class of entire analytic functions of order $\rho \leq 2$ and type $0 \leq K \leq \infty$ (including the limiting values of entire functions for $K = \infty$).

We have considered a realization of the representations $D(\lambda)$ by functions even with respect to the substitution $z \to -z$. In a completely analogous way one may construct a realization by odd entire functions. We note only that the canonical basis of this realization is the system of functions

$$f_n^{(\lambda)}(z) = \frac{i^n}{2^n} \sqrt{\frac{\Gamma(-2\lambda)}{n!\,\Gamma(n-2\lambda)}}\, z^{2n+1},$$

orthonormal with respect to condition (7), with measure

$$\rho_\lambda(\nu) = \frac{1}{\pi\Gamma(-2\lambda)} \left(\frac{2}{\nu}\right)^{2\lambda+1} K_{2\lambda+1}(\nu).$$

The infinitesimal operators of the representation will be

$$L_3^{(\lambda)} = \frac{1}{2} z \frac{d}{dz} - \left(\lambda + \frac{1}{2}\right), \qquad L_+^{(\lambda)} = \frac{1}{2} z^2,$$

$$L_-^{(\lambda)} = -\frac{1}{2} \frac{d^2}{dz^2} + \frac{4\lambda+3}{2z} \frac{d}{dz} - \frac{4\lambda+3}{2z^2}.$$

For $\lambda = -3/4$, all formulas of this realization pass into the corresponding formulas of paper (3).

It is a great pleasure for me to express my sincere gratitude to Academician N. N. Bogolyubov and I. T. Todorov for remarks concerning the question considered, which have been taken into account by the author.

Proof correction note. For $\lambda \neq -m/2$, where m is a positive integer, the constructed representations $u \to T_\lambda(u)$, $u \in SU(2)$, are multivalued. The operators

T_λ realize an exact representation of the group $\widetilde{SU(2)}$ in the linear locally convex topological space Φ', formed by entire functions of order $\rho \leq 2$ and type $0 \leq K < \infty$. The group $\widetilde{SU(2)}$ is the convolution (analytic continuation with respect to the group parameters) of the well-known group $\widetilde{SU(1,1)}$, the universal covering group of the Lorentz group L_3. On Φ' the operators T_λ are unbounded.

Physico-Technical Institute
Academy of Sciences of the Ukrainian SSR

Received
21 XII 1966

REFERENCES

1. S. S. Savnikov, *Yadern. fiz.*, **2**, 570 (1965).
2. S. S. Savnikov, *Yadern. fiz.*, **4**, no. 4 (1966).
3. S. S. Savnikov, *Ukr. fiz. zhurn.*, **10**, 684 (1965); DAN, **172**, no. 1 (1967).
4. V. Bargmann, *Comm. Pure and Appl. Math.*, **14**, 187 (1961).
5. M. A. Naimark, *Linear Representations of the Lorentz Group*, Moscow, 1958.
6. I. S. Gradshteyn, I. M. Ryzhik, *Tables of Integrals, Sums, Series, and Products*, Moscow, 1962.
7. I. M. Gelfand, G. E. Shilov, *Generalized Functions*, vol. 1, Moscow, 1959.

Note: Figure translations are in progress. See original paper for figures.