The highest exponent and the critical exponent of certain differential equations in a Banach space

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Abstract

The content of the article is determined by the decisive role of a special exponent in the study of solutions to differential equations. The main result is contained in Theorem 1: If the operator $A(t)$ in equation

$$\frac{dx}{dt} = A(t)x \tag{1}$$

is such that for some number $\eta > 0$ there exists a sequence of numbers $\{h_j\} \to \infty$ as $j \to \infty$, such that in each segment of length h_j there is at least one value τ_j for which

$$\|T_{\tau_j} A(t) - A(t)\| \le \exp(-\eta h_j), \tag{2}$$
$$T_{\tau_j} A(t) = A(t + \tau_j),$$

, then the Lyapunov σ_s and special σ^* exponents of equation (1) coincide. Theorem 2 illustrates the application of the obtained results to the study of the uniform and asymptotic stability of the zero solution of a certain differential equation in a Hilbert space. Bibliography: 4 items.

Full Text

Preamble

In this section, we consider the linear differential equation in a Banach space E:

$$\frac{dx(t)}{dt} = A(t)x(t), \quad t \in [0, +\infty) \tag{0.1}$$

where $A(t)$ is a bounded linear operator such that $\sup_{t\ge 0} \|A(t)\| \le M$. Let $U(t, s)$ be the evolution operator (propagator) for equation (0.1), satisfying:

$$\frac{dU(t, s)}{dt} = A(t)U(t, s), \quad U(s, s) = I \tag{0.4}$$

As established in [1], the upper and lower Lyapunov exponents for the system (0.1) can be characterized by the growth rates of the norm of the evolution operator. Specifically, the upper exponent α^* is defined as:

$$\alpha^* = \lim_{t-s\to\infty} \frac{\ln \|U(t,s)\|}{t-s} \tag{0.6}$$

where the limit is taken such that $0 \leq s \leq t < +\infty$. It is known that for any $\epsilon > 0$, there exists a constant N_ϵ such that $\|U(t,s)\| \leq N_\epsilon \exp[(\alpha^* + \epsilon)(t-s)]$.

1. Stability and Growth Estimates

Let α_s denote the Bohl exponent. For any $\beta > 0$ and all $t > 0$, the following inequality holds:

$$\|U(t,0)\| \leq N_\beta \exp[(\alpha_s + \beta)t] \tag{1.1}$$

If we consider a solution $x(t)$ with initial condition $x_0 \in E$, we can analyze the behavior of the system in various domains D. Under the assumption that $\rho < 0$, the system exhibits stability. Furthermore, if there exists $A_0 > 0$ such that the conditions in (1.2) are satisfied for a non-empty set D_1, then according to the results in [1], we have the estimate:

$$\|U(t,s)\| \leq N \exp[(\alpha_s + \beta)(t-s)] \tag{1.3}$$

where N depends on the parameters of the operator $A(t)$. Conversely, for $s \in D_2$, the norm of the evolution operator satisfies the lower bound:

$$\|U(t,s)\| \geq \exp[(\alpha_s + \rho)(t-s)] \tag{1.4}$$

where $0 < \rho < \rho_0$.

2. Asymptotic Behavior and Perturbations

We assume that the operator $A(t)$ satisfies a condition of slow variation or regularity, such as:

$$\|T_{\chi_j} A(t) - A(t)\| \leq C \exp(-\gamma h_j) \tag{2.1}$$

where $T_{\chi_j} A(t) = A(t + \chi_j)$. Under these conditions, we investigate the relationship between the spectral properties of $A(t)$ and the exponents of the system. If $\rho_0 = 0$ in (2.2), then for $t - s > T$, the evolution operator satisfies:

$$\|U(t,s)\| \leq \exp\left[\alpha_s + \rho_0 + \epsilon\right](t-s) \tag{2.3}$$

As $j \to \infty$ and $t \to \infty$, we can show that for $t \in [l, l+L]$, the norm $\|U(t,s)\|$ is bounded from below by:

$$\|U(t,s)\| \geq \exp\left[\alpha_s + \beta_j - \epsilon\right](t-s) \tag{2.5}$$

By applying the translation operator T_{χ_j} and considering the identity (2.13):

$$T_{\chi_j}U(t,s) - U(t,s) = \int_s^t U(t,\tau)[T_{\chi_j}A(\tau) - A(\tau)]T_{\chi_j}U(\tau,s)d\tau \tag{2.13}$$

we can derive that the difference between the perturbed and original evolution operators vanishes asymptotically. Specifically, using the estimates (2.1) and (2.11), we obtain:

$$\lim_{j\to\infty} \|T_{\chi_j}U(t_j,\tau_j) - U(t_j,\tau_j)\| \exp(-\alpha_s t_j) = 0 \tag{2.15}$$

This implies that $\alpha_s = \alpha^*$, confirming that for operators satisfying the regularity condition (2.1), the Bohl and Lyapunov exponents coincide.

3. Numerical Ranges and Stability Criteria

Following the approach in [3], let us define the numerical range of the operator $A(t)$. Suppose there exist functions $\alpha(t)$ and $\rho(t)$ such that for all $\phi \in E$ with $\|\phi\| = 1$:

$$\alpha(t) \leq \mathrm{Re}(A(t)\phi,\phi) \leq \rho(t) \tag{3.2}$$

Then for any solution $x(t)$ of (3.1), the growth is constrained by:

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \alpha(s)ds \leq \lim_{t\to\infty} \frac{\ln\|x(t)\|}{t} \leq \lim_{t\to\infty} \frac{1}{t}\int_0^t \rho(s)ds \tag{$MATH_{0003}$}$$

If the condition $\int_0^\infty \rho(s)ds < 0$ is satisfied, the system (3.1) is asymptotically stable. In the context of the class $Z(\nu, N)$ defined in [1], the system is exponentially stable if there exist constants N and $\nu > 0$ such that:

$$\|x(t)\| \leq N\exp[-\nu(t-s)]\|x(s)\|$$

This stability is closely linked to the property $\alpha^* = \alpha_s$ under the perturbation conditions discussed above.

References

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4. Riesz, F., Sz.-Nagy, B., *Functional Analysis*, Ungar, New York, 1955.

Note: Figure translations are in progress. See original paper for figures.