A CRITERION FOR THE VALIDITY OF THE TRANSLATION THEOREM IN THE THEORY OF NORMAL ALGORITHMS

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Abstract

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A CRITERION FOR THE VALIDITY OF THE TRANSLATION THEOREM IN THE THEORY OF NORMAL ALGORITHMS

(Presented by Academician P. S. Novikov on 5 IV 1966)

We use the terminology adopted in ([1]). By an algorithm we shall everywhere mean a normal algorithm.

Let alphabets A and B be given. If to each letter ξ of A there is put in correspondence a word B_ξ in B, then the mapping $\mathfrak{T}$ of the alphabet A into the set of all words in the alphabet B so constructed will be called a **method of translation from the alphabet A into the alphabet B**. Let P be a word in A. Then the **translation of the word P under the method of translation $\mathfrak{T}$ from A into B** is the word in B equal to $B_{\xi_1} \dots B_{\xi_n}$, if $P \doteq \xi_1 \dots \xi_n$, and equal to Λ, if $P \doteq \Lambda$. The translation of the word P under $\mathfrak{T}$ is denoted by $\mathfrak{T}\lfloor P \rfloor$.

If $\mathfrak{A}$ is an algorithm in A, then the **translation of the algorithm $\mathfrak{A}$ under the method of translation $\mathfrak{T}$ from A into B** is the algorithm in B obtained as a result of simultaneously replacing all letters from A occurring in $\mathfrak{A}$ by their translations under $\mathfrak{T}$. The translation of the algorithm $\mathfrak{A}$ under $\mathfrak{T}$ is denoted by $\mathfrak{A}^{\mathfrak{T}}$. A method of translation from A into B will be called **degenerate** if A consists of one letter and this letter is translated into the empty word.

We shall agree to use the letters $\xi, \eta, \zeta, \gamma, \delta$ to denote letters of the alphabet A, the letters $L, M, \dots, R$ to denote words in A, the letters $E, F, G, H, S, T, \dots, Z$ to denote words in B, the letter $\mathfrak{A}$ to denote algorithms in A, and the letter $\mathfrak{T}$ to denote methods of translation from A into B.

A method of translation $\mathfrak{T}$ is called **one-to-one**, if

$$\forall P \forall Q (\mathfrak{T}\lfloor P \rfloor \doteq \mathfrak{T}\lfloor Q \rfloor \supset P \doteq Q).$$

A method of translation $\mathfrak{T}$ is called **admissible**, if

$$\forall \mathfrak{A} \forall P (\mathfrak{T}\lfloor \mathfrak{A}\lfloor P \rfloor \rfloor \simeq \mathfrak{A}^{\mathfrak{T}} \mathfrak{T}\lfloor P \rfloor),$$

i.e., if for $\mathfrak{T}$ the conclusion of the translation theorem in ([1]) is valid.

The present paper contains the solution of a problem posed to me by A. A. Markov, concerning the finding of an algorithm which recognizes the admissibility of methods of translation.

Theorem. *A method of translation $\mathfrak{T}$ from A into B is admissible if and only if*

$$\forall \xi\eta\zeta GH(B_\xi B_\eta \doteq GB_\zeta H \supset \zeta \doteq \xi \vee (\zeta \doteq \eta \& H \doteq \Lambda)) \tag{1}$$

and $\mathfrak{T}$ is nondegenerate.

The proof of this theorem is based on the following lemmas:

Lemma 1. *From the admissibility of a method of translation $\mathfrak{T}$ there follows its one-to-one character. In particular, from the admissibility of $\mathfrak{T}$ it follows that $\forall \xi\,(B_\xi \neq \Lambda)$ and that $\mathfrak{T}$ is nondegenerate.*

Proof. Let $\mathfrak{T}$ be admissible and $\mathfrak{T}\lfloor P \rfloor \doteq \mathfrak{T}\lfloor Q \rfloor$.

If $P \neq Q$, then, without loss of generality, one may assume that Q does not occur in P. Let $\mathfrak{A}$ be an algorithm with scheme $\{Q \to Q\cdot\}$. Then $\mathfrak{T}\lfloor \mathfrak{A}\lfloor P \rfloor \rfloor \doteq \mathfrak{T}\lfloor P \rfloor$ and $\neg\, !\, \mathfrak{A}^{\mathfrak{T}}\mathfrak{T}\lfloor P \rfloor$, which contradicts the admissibility of $\mathfrak{T}$. Hence $P \doteq Q$.

Lemma 2. *From the admissibility of $\mathfrak{T}$ it follows that*

$$\forall \xi\eta\zeta GH(B_\xi B_\eta \doteq GB_\zeta H \supset \xi \doteq \xi \vee \zeta \doteq \eta).$$

Proof. Let $B_\xi B_\eta \mathrel{\mathring{=}} GB_\nu H$ and let $\mathfrak{A}$ be an algorithm with scheme $\{\xi \to \cdot\xi\eta\xi\}$. Then, if $\xi \neq \eta$, we have $\mathfrak{L}\mathfrak{A}_{\mathfrak{L}}\xi\eta_{\rfloor\rfloor} \mathrel{\mathring{=}} B_\xi B_\eta$ and $\mathfrak{A}^{\xi}_{\mathfrak{L}}\mathfrak{L}\xi\eta_{\rfloor\rfloor} \mathrel{\mathring{=}} G_1 B_\xi B_\eta B_\xi H_1$, where G_1 and H_1 are the wings of the first occurrence of B_ξ in $B_\xi B_\eta$. Since $B_\xi \neq \Lambda$, this contradicts the admissibility of $\mathfrak{S}$. Hence,

$$\xi \mathrel{\mathring{=}} \eta \vee \xi \mathrel{\mathring{=}} \eta.$$

From Propositions 1.1.3 and 1.1.4 of [4] it follows that

Lemma 3. *Let $F \neq \Lambda$; then*

$$\exists S \forall X(XF \mathrel{\mathring{=}} FX \supset \exists k\,(X \mathrel{\mathring{=}} S^k)).$$

Lemma 4. *From the admissibility of $\mathfrak{S}$ it follows that*

$$\forall \xi\eta GH\,(B_\xi B_\eta \mathrel{\mathring{=}} GB_\eta H \supset \eta \mathrel{\mathring{=}} \xi \vee H \mathrel{\mathring{=}} \Lambda).$$

Proof. Let

$$B_\xi B_\eta \mathrel{\mathring{=}} GB_\eta H \mathrel{\mathring{=}} G_1 B_\eta H_1,$$

where G_1 and H_1 are the wings of the first occurrence of B_η in $B_\xi B_\eta$, and let $\eta \neq \xi$. For any word R consider the algorithm $\mathfrak{A}_R$ with scheme $\{\eta \to \cdot R\}$. Then

$$\mathfrak{L}\mathfrak{A}_{R\mathfrak{L}}\xi\eta_{\rfloor\rfloor} \mathrel{\mathring{=}} B_\xi \mathfrak{L}R_\rfloor, \qquad \mathfrak{A}^{\xi}_{R\mathfrak{L}}\mathfrak{L}\xi\eta_{\rfloor\rfloor} \mathrel{\mathring{=}} G_1 \mathfrak{L}R_\rfloor H_1.$$

By the admissibility of $\mathfrak{S}$ we have

$$B_\xi \mathfrak{L}R_\rfloor \mathrel{\overset{\circ}{=}} G_1 \mathfrak{L}R_\rfloor H_1$$

for every R. For $R \mathrel{\overset{\circ}{=}} \Lambda$ we obtain

$$B_\xi \mathrel{\overset{\circ}{=}} G_1 H_1.$$

We have

$$\forall R\,(H_1 \mathfrak{L}R_\rfloor \mathrel{\overset{\circ}{=}} \mathfrak{L}R_\rfloor H_1).$$

If $H_1 \neq \Lambda$, then, by Lemma 3,

$$\exists S \forall R \exists k\,(\mathfrak{L}R_\rfloor \mathrel{\overset{\circ}{=}} S^k).$$

Taking as R the letter ξ, and then the letter η, we obtain

$$\exists S \exists k_1 \exists k_2\,(B_\xi \mathrel{\overset{\circ}{=}} S^{k_1} \;\&\; B_\eta \mathrel{\overset{\circ}{=}} S^{k_2}).$$

Now we have

$$\mathfrak{L}\xi\eta_\rfloor \mathrel{\overset{\circ}{=}} B_\xi B_\eta \mathrel{\overset{\circ}{=}} S^{k_1+k_2} \mathrel{\overset{\circ}{=}} S^{k_2+k_1} \mathrel{\overset{\circ}{=}} B_\eta B_\xi \mathrel{\overset{\circ}{=}} \mathfrak{L}\eta\xi_\rfloor \quad \text{and} \quad \xi\eta \neq \eta\xi$$

(since $\eta \neq \xi$). But this contradicts mutual uniqueness (see Lemma 1). Hence

$$H_1 \mathrel{\overset{\circ}{=}} \Lambda,$$

and consequently

$$H \mathrel{\overset{\circ}{=}} \Lambda.$$

From Lemmas 1, 2, and 4 it follows that the nondegeneracy of $\mathfrak{S}$ and condition (1) are necessary for the admissibility of $\mathfrak{S}$. The following Lemmas 5–8 are obvious.

Lemma 5. *From (1) it follows that*

$$\forall \xi\eta EF\,(B_\eta \mathrel{\overset{\circ}{=}} EB_\xi F \supset \eta \mathrel{\overset{\circ}{=}} \xi). \tag{2}$$

In particular, from (1) it follows that

$$\forall \xi\eta F\,(B_\eta \mathrel{\overset{\circ}{=}} B_\xi F \supset \eta \mathrel{\overset{\circ}{=}} \xi), \tag{3}$$

$$\forall \xi\eta E\,(B_\eta \mathrel{\overset{\circ}{=}} EB_\xi \supset \eta \mathrel{\overset{\circ}{=}} \xi). \tag{4}$$

Lemma 6. *If $\mathfrak{S}$ is nondegenerate and satisfies condition (3) or (4), then it is mutually unique and, in particular,*

$$\forall \xi\,(B_\xi \neq \Lambda).$$

Lemma 7. a) *Let $\mathfrak{S}$ satisfy (4); then, if*

$$\mathfrak{L}Q_\rfloor \mathrel{\overset{\circ}{=}} E\mathfrak{L}R_\rfloor,$$

then E is a translation of some word.

b) *Let $\mathfrak{S}$ satisfy (3); then, if*

$$\mathfrak{L}Q_\rfloor \mathrel{\mathring{\doteq}} \mathfrak{L}R_\rfloor F,$$

then F is a translation of some word.

Lemma 8. *Let $\mathfrak{S}$ be an arbitrary method of translation and let*

$$EF \mathrel{\mathring{\doteq}} \mathfrak{L}Q_\rfloor, \qquad Q \neq \Lambda.$$

Then there exist L, ξ, M, S, T such that

$$L\xi M \mathrel{\mathring{\doteq}} Q, \qquad ST \mathrel{\mathring{\doteq}} B_\xi, \qquad E \mathrel{\mathring{\doteq}} \mathfrak{L}L_\rfloor S, \qquad F \mathrel{\mathring{\doteq}} T\mathfrak{L}M_\rfloor, \qquad T \neq \Lambda,$$

and, moreover, if E is not a translation of any word, then $S \neq \Lambda$.

Lemma 9. *Let $\mathfrak{S}$ satisfy (1). Then, if*

$$X\mathfrak{L}P_\rfloor Y \mathrel{\mathring{\doteq}} \mathfrak{L}Q_\rfloor$$

and

$$\neg\exists N\,(X \mathrel{\mathring{\doteq}} \mathfrak{L}N_\rfloor),$$

then: a) P is a power of some letter ξ:

$$P \mathrel{\mathring{\doteq}} \xi^k, \qquad k = [P]^\partial;$$

b) $X * \mathfrak{L}P_\rfloor * Y$ is not the first occurrence of $\mathfrak{L}P_\rfloor$ in $\mathfrak{L}Q_\rfloor$.

Proof. We apply induction on $[P]^\partial$.

Basis. For $[P]^\partial = 0$ we have

$$P \mathrel{\mathring{\doteq}} \Lambda,$$

and the assertion of the lemma is obvious.

Inductive step. $[P^\circ \geq 1]$. First consider the case $[P^\circ \geq 2]$. There exist ξ and P_1 such that $P \doteq \xi P_1$. We have $\mathfrak{L}[Q_\perp \doteq XB_\xi\mathfrak{L}[P_1]Y$. By Lemmas 5 and 7a), $\neg\mathfrak{N}_1(XB_\xi \doteq \mathfrak{L}[N_{1\perp}])$. Applying the induction hypothesis, we find γ such that $P_1 \doteq \gamma^{k-1}$, $P \doteq \xi\gamma^{k-1}$.

Obviously, $Q \neq \Lambda$. Taking X and $\mathfrak{L}[P]Y$ as E and F in Lemma 8, we find L, ξ, M, S, T such that $Q \doteq L\xi M$, $B_\xi \doteq ST$, $X \doteq \mathfrak{L}[L]S$, $\mathfrak{L}[P]Y \doteq T\mathfrak{L}[M_\perp]$, $T \neq \Lambda$, $S \neq \Lambda$. We obtain $B_\xi B_\gamma^{k-1}Y \doteq T\mathfrak{L}[M_\perp]$.

Hence, by Lemma 5, we find U such that $B_\gamma \doteq TU$, $U \neq \Lambda$. We have $UB_\gamma^{k-1}Y \doteq \mathfrak{L}[M_\perp]$, $M \neq \Lambda$. Let $M \doteq \eta M_1$. Then $UB_\gamma^{k-1}Y \doteq B_\eta\mathfrak{L}[M_{1\perp}]$. Hence, by Lemma 5, we find V such that $B_\eta \doteq UV$, $V \neq \Lambda$. We have $B_\gamma^{k-1}Y \doteq V\mathfrak{L}[M_{1\perp}]$. Hence, by Lemma 5, we find W such that $B_\gamma \doteq VW$, $W \neq \Lambda$. We have $WB_\gamma^{k-2}Y \doteq \mathfrak{L}[M_{1\perp}]$, $M_1 \neq \Lambda$. Let $M_1 \doteq \delta M_2$. Then $WB_\gamma^{k-2}Y \doteq B_\delta\mathfrak{L}[M_{2\perp}]$. As before, we find Z such that $B_\delta \doteq WZ$, $Z \neq \Lambda$. We now obtain $B_\xi B_\eta \doteq SB_\xi V$,

$B_\xi B_\gamma \doteq TB_\eta W$, $B_\eta B_\delta \doteq UB_\gamma Z$. By (1) it follows that $\xi \doteq \xi \doteq \eta \doteq \gamma$, $ST \doteq TU \doteq UV$, $S \doteq U$, $P \doteq \xi^k$, $\mathfrak{L}[P_\perp] \doteq (ST)^k \doteq (TS)^k$, $\mathfrak{L}[Q_\perp] \doteq \mathfrak{L}[L]S\mathfrak{L}[P]Y \doteq \mathfrak{L}[L]S(TS)^{kY} \doteq \mathfrak{L}[L](ST)^{kSY} \doteq \mathfrak{L}[L]\mathfrak{L}[P]SY$. Since $S \neq \Lambda$, the occurrence $X * \mathfrak{L}[P_\perp] * Y$, equal to $\mathfrak{L}[L]S * \mathfrak{L}[P_\perp] * Y$, is not the first.

Following the pattern of this proof, it is not difficult to construct also the proof for the case $[P^\circ = 1]$.

Lemma 10. Let $\mathfrak{T}$ satisfy (1). Then, if $X * \mathfrak{T}P * Y$ is the first occurrence of $\mathfrak{T}[P_\perp]$ in $\mathfrak{T}[Q_\perp]$, then X and Y are translations of certain words under $\mathfrak{T}$.

This lemma follows from Lemmas 5, 7, 9. To prove that the admissibility of a translation method follows from condition (1) and nondegeneracy, it remains to use Lemmas 5, 6, 10 and carry out the same arguments as in (1) in the proof of the translation theorem.

Remark. The author has proved that a criterion for the property of translation methods analogous to admissibility for Post canonical calculi (see ([2])), as well as for Thue semisystems (see ([3]), § 71), is the nondegeneracy of the translation method and the property

$$\forall \xi\eta\xi GH\, (B_\xi B_\eta \doteq GB_\xi H \supset ((\xi \doteq \xi \,\&\, G \doteq \Lambda) \vee (\xi \doteq \eta \,\&\, H \doteq \Lambda))),$$

and for normal Post calculi (see ([2])) it is the nondegeneracy of the translation method and property (3).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.