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Abstract

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ON THE PROBLEM WITH AN OBLIQUE DERIVATIVE IN THREE-DIMENSIONAL SPACE

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1. Let D be a bounded three-dimensional domain of class C^3. On its boundary ∂D there is given a vector field $l(x)$ $(x \in \partial D)$ of class C^3, not vanishing at 0. Denote by M the set of points at which $l(x)$ lies in the tangent plane T_x to ∂D.

We assume that: 1) M consists of a finite number of closed curves of class C^2; 2) if a curve $\gamma \subset M$, then $l(x)$ is not tangent to γ at any point of γ. Let

$$L = \sum_{i,j=1}^{3} a_{ij}\, \partial^2/\partial x_i \partial x_j + \sum_{i=1}^{3} b_i\, \partial/\partial x_i$$

be a nondegenerate elliptic operator whose coefficients belong to $C^3(\overline{D})$, where $\overline{D} = D \cup \partial D$. Consider the following boundary-value problem A: $LF = f$ in D, $f \in C^3(\overline{D})$, and $dF/dl = 0$ on $\partial D \setminus M$. By a solution of it we shall mean a bounded function $F(x)$, smooth in D, such that $LF = f$ in D, $F \in C^1(\overline{D} \setminus M)$ and $dF/dl = 0$ for $x \in \partial D \setminus M$.

In the paper it is shown that the solution of problem A is uniquely determined by its limiting values on a certain part of the set M. An existence theorem for a solution of problem A is also proved for a broad class of boundary values.

For some special cases, smooth solutions of the problem with an oblique derivative were studied by A. V. Bitsadze and A. Janashiauskas. When the present paper was being submitted for publication, I became aware of the general results of the work of Yu. V. Egorov and V. A. Kondrat' ev $(^2)$ on the problem with an oblique derivative. But they too are restricted to finding smooth solutions of problem A, which in the general case does not make it possible to obtain existence theorems. The question of selecting, among the solutions of problem A that are continuous in D and smooth in D, remains open.

The three-dimensional case is considered only for simplicity; analogous arguments can also be carried out for a space of n dimensions.

2. Let $n(x)$ be the vector of the inner normal to ∂D at the point x. Introduce the vector field $l_1(x) = l(x)$, if $(l(x), n(x)) > 0$; $l_1(x) = -l(x)$ in the opposite case.

Let M_δ be the δ-neighborhood of M and $S_\delta = M_\delta \cap \partial D$. If δ is sufficiently small, then S_δ consists of disjoint neighborhoods γ_δ of the curves γ (relative to ∂D), and the domain γ_δ is divided by the curve γ into parts γ_δ^i $(i = 1, 2)$. We split each point a of the curve γ into points a^i $(i = 1, 2)$ and shall call the ε-neighborhood $(\varepsilon < \delta)$ of the point a^i the set of points from γ_δ^i that are removed from the point a by a distance less than ε. We shall denote by

$$\lim_{x \to a} \Phi(x)$$

or simply by $\Phi(a^i)$ the limiting value of the function $\Phi(x)$ at the point a^i. The collection of points a^i $(a \in \gamma)$ will be called the side γ^i of the curve γ. Let $n_i(a)$ be the vector of the normal* to γ at the point a, lying in the plane T_a and directed toward γ_δ^i. From condition 2) it follows that the scalar product* $(l_1(a^i), n^i(a)) \neq 0$, and the sign of the scalar product is constant for all points of the side γ^i. We shall call the side γ^i of the curve γ **positive** (respectively **negative**) if the scalar product is negative (positive). If both sides of γ^i are positive, then γ is called **positive** (**negative**); if the sides of γ^i have different signs, then it is called **zero**. Let us renumber the special curves and denote by γ_k^i the i-th side of the k-th special curve. The set of points of the positive sides will be denoted by M_+.

The homogeneous problem A $(f = 0)$ will be called problem A_0.

Theorem. Suppose M_+ is nonempty. Assign to each positive side γ_k^i a continuous function $\varphi_k^i(a)$, $a \in \gamma_k$. There exists a unique solution $F(x)$ of problem A such that, for all $\alpha^i \in \gamma_k^i$,

$$\lim_{x \to \alpha^i} F(x) = \varphi_k^i(\alpha).$$

If M_+ is empty, then the solutions of problem A_0 are only constants.*

Remark. It follows easily from the theorem that if for all points α of a positive curve γ_k one has $\varphi_k^1(\alpha) = \varphi_k^2(\alpha)$, then the solution $F(x)$ is continuous in a three-dimensional neighborhood of γ_k. The solution is discontinuous, generally speaking, at each point of the negative curves, even if the φ_k^i are arbitrarily smooth functions.

3. Let $U \subset D$ be a smooth domain, and on a closed set G with smooth boundary ∂G (in the topology of ∂U) let there be given a smooth vector field $\lambda(x)$, not

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* In the natural metric defined by the operator L.

lying in the tangent plane to ∂U. We shall say that a function $\Phi(x)$ is a solution of the **mixed boundary-value problem** C for the domain U if $L\Phi = f$ in U, and on $(\partial U \setminus G) \cup \partial G$ the Dirichlet condition $\Phi(x) = \varphi(x)$ is prescribed, while on $G \setminus \partial G$,

$$d\Phi/d\lambda = 0.$$

The theorem on existence and uniqueness of the solution of problem C for a sufficiently smooth function φ follows, for example, from (5). By passage to the limit one can prove the theorem of existence and uniqueness for more general boundary values φ, in particular for discontinuous boundary functions φ, used below. The proof of the theorem is obtained by passage to the limit from problem C for the domain ∂D, in which $\bar{S}_\delta$ is taken as G. Let us first consider problem A_0.

Let

$$U = D \cap \{x : \|x - a\| < \varepsilon\},$$

where a_1 is a point of the curve γ having the positive side γ^1, and let

$$\Gamma_\delta = \gamma_\delta^1 \cap \partial U \qquad (\delta < \varepsilon).$$

Consider the solution U_δ of problem C_0 (i.e., problem C with $f = 0$) for the domain U with the following boundary conditions:

$$U_\delta = 1 \text{ on } \Gamma_\delta; \qquad U_\delta = 0 \text{ on } \partial U \setminus \gamma_\varepsilon^1; \qquad du_\delta/dl = 0 \text{ on } \gamma_\varepsilon^1 \setminus \Gamma_\delta.$$

As $\delta \to 0$ the functions u_δ decrease. This follows at once from the following lemma (see (3), p. 14).

Lemma 1. If $L\Phi = 0$, the function $\Phi(x)$ $(x \in U)$ is not constant, attains a minimum at a point $\xi \in \partial U$, and in some neighborhood of the point ξ is differentiable, then for any vector l forming an acute angle with the inner normal at ξ,

$$d\Phi/dl > 0$$

at the point ξ.

Denote

$$u_0 = \lim_{\delta \to 0} u_\delta.$$

Lemma 2. $u_0(\alpha^1) = 1$.

Let us note that it is enough to prove Lemma 2 for an arbitrarily small domain U. In its proof a device close to the known method of "freezing" the coefficients is used, thanks to which the problem is essentially reduced to a two-dimensional one and is solved analogously to the two-dimensional case (see (1), proof of Theorem 1).

Let Δ be a segment of the curve γ having the positive side γ^1, and

$$\Delta^1 = \{\alpha^1;\ \alpha \in \Delta\}.$$

From Lemma 2 follows the existence of a solution F_Δ of problem A_0 satisfying the conditions: the function $F_\Delta(\alpha)$ is equal to 0 on $M_+ \setminus \Delta^1$, and $F_\Delta(\alpha^1) = 1$, if α^1 is an interior point of Δ^1. Indeed, let μ_Δ be the set of points of γ_δ^1 whose distance to the segment Δ is less than to $M_+ \setminus \Delta$. Consider the solution Φ_δ of problem C_0 for the domain ∂D with conditions:

$$\Phi_\delta = 1 \text{ on } \mu_\Delta; \qquad \Phi_\delta = 0 \text{ on } S_\delta - \mu_\Delta.$$

It is clear that inside a sufficiently small—

* If M_+ is empty, then the solution of problem A exists under an additional condition on the right-hand side of the same type as in the Neumann problem.

neighborhood U of an interior point $a \in \Delta$, the solution Φ_δ will be greater than the solution constructed in Lemma 2, and inside a sufficiently small neighborhood W of the point $\beta \in M_+ \setminus \Delta^1$ it will be less than $1 - w_\delta$, where w_δ is the solution constructed in Lemma 2 for W.

Therefore, contracting first μ_Δ to the segment Δ, and then $S_\delta \setminus \mu_\Delta$ to $M \setminus \Delta^1$, we obtain a function F_Δ with the required boundary values, satisfying, by virtue of the monotonicity of passage to the limit, the equation $LF_\Delta = 0$ in D. It is not difficult to prove that F_Δ also satisfies the condition $dF_\Delta/dl = 0$ on $\partial D \setminus M$.

Using the functions F_Δ, it is easy to prove, by means of the same monotone limiting passage, that there exists a solution of problem A_0 with arbitrary continuous boundary values*.

4. Consider problem C_δ with the same boundary conditions as in Lemma 2, but now let γ^1 be the negative side. Denote the corresponding solution by v_δ.

Lemma 3. $v_0 = \lim_{\delta\to 0} v_\delta \equiv 0$.

If the field of conormal vectors to ∂D is taken as the vector field $l(x)$, then this result is well known. We shall prove that the solution v_δ in the case of the negative side γ^1 is less than the corresponding solution v_δ^1 for the same domain, but for conormal vectors.

It is easy to verify that $d(v_\delta^1 - v_\delta)/dl \leq 0$, if the domain U is chosen sufficiently small. Therefore, by Lemma 1, the minimum of $v_\delta^1 - v_\delta$ is attained on $\Gamma_\delta \cup (\partial U - \gamma_\varepsilon^1)$, where the function $v_\delta^1 - v_\delta$ is equal to 0 by assumption.

An even stronger result is true. Let again γ^1 be the negative side and let w_δ be the solution of problem C_0 with boundary conditions $w_\delta = 1$ on $\Gamma_\delta \cup (\gamma_\varepsilon^2 \cap \partial U)$; $w_\delta = 0$ on $\partial U \setminus (\gamma_\varepsilon^1 \cup \gamma_\varepsilon^2)$; $dw_\delta/dl = 0$ on $(\gamma_\varepsilon^1 \cap \partial U) \setminus \Gamma^\delta$.

Let $w_0 = \lim_{\delta\to 0} w_\delta$.

Lemma 4. $\lim\limits_{x\to a^1} w_0(x) < 1$.

Let the domain U and the field l_1 be the same as in Lemma 3, but let arbitrary smooth functions be prescribed on Γ_δ and $\partial U \setminus \gamma_\varepsilon^1$. Then the solution of the Γ_δ-problem C_0 with such boundary conditions converges, as $\delta \to 0$, to a function $F(x)$, and, by Lemma 3, the extremum of the function $F(x)$ is attained on $\partial U \setminus \gamma_\varepsilon^1$. Hence, by Lemma 1, the maximum principle follows: if the solution $F(x)$ of problem A_0 is continuous in M_+, then the extremum of $F(x)$ is attained on M_+. From this maximum principle the uniqueness of the solution of problem A follows at once.

Using the results of the next item, one can prove a stronger formulation of the maximum principle. Let $F(x)$ be a function bounded from below, satisfying in D the equation $LF = 0$, $dF/dl = 0$ on $\partial D \setminus M$, and $\lim_{x\to M_+} F(x) \geq 0$. Then $F(x) \geq 0$.

5. In proving the existence of a particular solution of the nonhomogeneous problem A, considerations based on the probabilistic interpretation of the problem with oblique derivative are used. We start from the diffusion process X^δ, constructed by M. I. Freidlin in (5), with reflection along the field $l(x)$, governed inside D by the operator L, and terminated at the moment ξ^δ of hitting S_δ. In the book (4) a method is given for constructing such a process X, such that the trajectories of X^δ coincide with the trajectories of X up to the moment ζ^δ, with $\zeta^\delta \uparrow \zeta$, where ζ is the termination time of the process X.

Lemma 1, in its probabilistic interpretation, means that the probability of the trajectory X_t of the process X hitting the curve γ, starting from the point $x \in \overline{U}$ before exit to $\partial U \setminus \gamma_\varepsilon^1$, tends to 1 if $x \to a^1$. Indeed, the function u_δ is the probability of hitting Γ_δ, starting from x, before exit to $\partial U \setminus \gamma_\varepsilon^1$.

* In this way one can obtain a solution of problem A_0 for a measurable bounded function as the "boundary values," but, as in the Dirichlet problem, the question of the behavior of the solution near the boundary is more complicated.

It is not difficult to prove that the mathematical expectation m_δ of the time τ_δ of reaching $\Gamma_\delta \cup (\partial U \setminus \gamma^{\varepsilon_1})$ is bounded uniformly in δ.

Therefore the time $\tau = \lim_{\delta\to 0} \tau_\delta$ is finite with probability 1, and the trajectory up to the time τ is a compact set. Since, with probability $u_0(x) = \lim_{\delta\to 0} u_\delta(x)$, the trajectory falls into any of the sets Γ_δ, it follows, by the compactness of the trajectory, that with probability $u_0(x)$ the trajectory x_t falls on γ. Let now $\zeta_u = \tau$ if the trajectory has reached γ, and otherwise $\zeta_u = \infty$. From Lemma 1 it follows that, for some $T > 0$,

$$\lim_{x \to \alpha^1} P_x\{\zeta_u < T\}\beta > 0.$$

Hence it is easily derived that the mathematical expectation of ζ is bounded over all initial points from $\overline{D}$. Further, just as in (5), it is proved that if $f(x) \in C^3(\overline{D})$, then

$$F(x) = M_x \int_0^\zeta f(x_t)\, dt$$

satisfies the equation

$$LF = -f$$

and the boundary conditions $dF/dl = 0$, while $F(x)$ tends to 0 when the point x approaches M_+.

Thus, a particular solution of problem A has been constructed.

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Note: Figure translations are in progress. See original paper for figures.