The completeness of certain function systems

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Abstract

Linearly independent systems

$$\{\ln r(x_i, y)\}, \quad \left\{\frac{\partial}{\partial n_y} \ln r(x_i, y)\right\} \quad (i = 1, 2, \dots), \tag{1}$$

are considered, where x_i are uniformly distributed on the circle S_1, and $y \in S$; here, S and S_1 are concentric circles. It is proved in this paper that the systems (1) are not strongly minimal.

Systems (1) are also considered for $x_i \in S$ and $y \in S$, where S is a piecewise smooth closed curve. It is proved that if x_i are located everywhere densely on S, then the systems (1) of discontinuous potential functions (where an arbitrary non-zero constant must be added to the second system in (1)) are linearly independent and complete in $L_2(S)$.

Regarding the system

$$\left\{\frac{1}{r(x_i, y)}\right\} \quad (i = 1, 2, \dots), \tag{2}$$

where $x_i \in S$, $y \in S$, and S is a closed surface of spatial discontinuous potential functions, it is proved that with an everywhere dense arrangement of points x_i on S, the system (2) is closed in $L_p(S)$ for $p = 2-\alpha$ for any $\alpha > 0$. Consequently, it is complete in $L_{p'}(S)$, where $p' = \frac{2-\alpha}{1-\alpha}$.

Bibliography: 10.

Full Text

Introduction

This work, published in 1967 (Volume III, No. 10), builds upon the foundational research of M. A. Aleksidze [2] and S. N. Kupradze [3]. We consider the application of numerical methods to boundary value problems, specifically focusing on the approximation of solutions using fundamental solutions of the Laplace equation. Following the methodologies established in [5-7], we define a set of points s_i and x_i distributed on the boundary S.

The analysis involves the discretization of the integral operator where the kernel is defined by the logarithmic potential $\ln r(x, y)$. For a system of $2N$ points, we define the coefficients a_s based on the distance between points x_i and y. Specifically, for $s = 2, 3, \ldots, N$, we utilize the relationship $a_s = a_{n+s-2}$, where $n = 2N$. The integral representation is given by:

$$a_s = \int \ln r(x_i, y) \ln r(x_{j+s-1}, y)\, ds_y$$

According to [8], the eigenvalues of the resulting circulant matrix can be expressed as:

$$\lambda_k = \sum a_i \omega^{i(k-1)}$$

where $\omega = \cos(\frac{2\pi}{n}) + i \sin(\frac{2\pi}{n})$. For $k = N$ in equation (2), the difference between coefficients $a_m - a_j$ is evaluated through the integral of the logarithmic kernel over the boundary S.

Mathematical Formulation and Convergence

We examine the properties of the operator in the space $L_2(S)$. As demonstrated in [5-7], the distribution of points x_i plays a critical role in the convergence of the approximation. The imaginary part of the approximate solution is related to the integral:

$$\text{Im}(\lambda_M) = \int \ln r(x_i, y) \phi(y) \ln r(x_j, y)\, ds_y$$

where $\phi(y) \in L_2(S)$. The kernel $k(x, y)$ of the integral equation (8) satisfies the condition:

$$\left\{ \int |\text{grad}_x k(x, y)| [r(x, y)]^{1-\alpha} \right\}^p ds_y < \infty$$

This ensures that the operator maps L_p to L_q effectively. Given the properties of the gradient of the logarithmic potential, the solution remains stable within the specified Sobolev or Lipschitz spaces.

Numerical Implementation and Results

The boundary value problem for the Laplace equation $\Delta u = 0$ in the domain G is solved by representing the potential as a linear combination of fundamental solutions. The function $\psi(y)$ is defined as:

$$\psi(y) = \phi(y) - 2\alpha \sum \ln r(x_k, y) \rho(y)$$

The angular components are handled using the arctg function to represent the argument $\text{Arg}(z - a_k)$. As noted in [5-7] and equation (10), the approximation error depends on the density of the points x_i on the boundary.

For $p = 2 - \alpha$ (where $\alpha > 0$), the kernel $k(x, y)$ belongs to the space $L_{p'}(S)$. If the density function $\rho(y)$ vanishes, the system reduces to a simpler form. The convergence in L_p spaces is guaranteed by the regularity of the boundary S and the properties of the chosen basis functions.

References

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Note: Figure translations are in progress. See original paper for figures.