EXISTENCE AND UNIQUENESS THEOREMS FOR SOLUTIONS OF HAMMERSTEIN EQUATIONS

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Abstract

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EXISTENCE AND UNIQUENESS THEOREMS FOR SOLUTIONS OF HAMMERSTEIN EQUATIONS

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The paper considers the nonlinear Hammerstein integral equation

$$x(t) = \int_\Omega k(t,s) f[s, x(s)]\, ds + h(t). \tag{1}$$

Here Ω is a bounded closed set of a finite-dimensional space; $f(s,u)$ is an operator satisfying the Carathéodory conditions and acting from $\Omega \times R^n$ into R^n; $k(t,s)$ $(t, s \in \Omega)$ is a symmetric (i.e., $k(s,t) = [k(t,s)]^*$) matrix, measurable in the aggregate of the variables; R^n is the real n-dimensional space. Such equations have been studied by methods of functional analysis in ([1–8]); the principal constructions in this connection were carried out in spaces of vector-functions C, $\mathcal{L}_p$, and in Orlicz spaces. In the present note the investigation of equations (1) is carried out in general functional spaces, whose theory is set forth, for example, in ([9]). This approach makes it possible to formulate several simple assertions on the solvability of equation (1), which, when applied to concrete spaces, contain a large part of the previously known results. The transition to general spaces has already made it possible, for scalar equations, to prove finer results, new even for the spaces $\mathcal{L}_p$.

The theorems presented in the present article reduce the investigation of Hammerstein equations to the study, in various spaces, of the linear integral operator K with matrix-kernel $k(t,s)$ and of the superposition operator $fx(s) = f[s, x(s)]$. Various theorems in this direction are set forth, for example, in ([9–11]).

Let us also note that the results of the article are naturally carried over to equations with a Lebesgue integral with respect to an arbitrary measure, in particular, to infinite systems; in this case, instead of Banach spaces, one may consider locally convex spaces.

1. Denote by S the space of measurable almost everywhere finite vector-functions on Ω with values in R^n. Put

$$\langle x, y\rangle = \int_\Omega (x(s), y(s))\, ds. \tag{2}$$

A Banach space E of vector-functions from S is called **ideal** if from $|x| \le |y|$, $x \in S$, $y \in E$, it follows that $x \in E$ and $\|x\|_E \le \|y\|_E$ (by $|x|$ is denoted the vector whose components are equal to the moduli of the components x; inequalities for vectors are understood componentwise). Denote by E^0 the totality of elements of E with absolutely continuous norm.

Every ideal space of functions with values in R^n may be regarded as the direct sum of n spaces $E_1, \dots, E_n$ of scalar functions. Let Ω_i $(i = 1, \dots, n)$ be the supports of E_i, i.e., such subsets of Ω that every function from E_i vanishes outside Ω_i, while in E_i there exist functions positive for $s \in \Omega_i$.

We shall call ideal spaces E and F **dual** if

$$\|x\|_E = \sup_{\|y\|_F \leqslant 1} \langle x, y\rangle, \qquad \|y\|_F = \sup_{\|x\|_E \leqslant 1} \langle x, y\rangle. \tag{3}$$

For each ideal space E, by E' we shall denote the space of vector-functions whose components vanish outside the supports Ω_i of the spaces E_i, and for which the norm

$$\|y\|_{E'} = \sup_{\|x\|_E \leqslant 1} \langle x, y\rangle \tag{4}$$

is meaningful.

The spaces E and E' are dual if and only if, for any sequence $x_m \in E$ converging in measure to $x \in E$, the inequality

$$\|x\|_E \leqslant \lim_{m\to\infty} \|x_m\|_E$$

holds.

Among ideal spaces are the space E_{u_0} (u_0 is a nonnegative function from S) of vector-functions for which the norm

$$\|x\|_{E_{u_0}} = \inf\{\lambda : |x| \leqslant \lambda u_0\}, \tag{5}$$

is meaningful; the space E'_{u_0} dual to it; the space $\mathcal{L}_p$; the Orlicz spaces, and many others.

Let $E_1, \dots, E_m, X$ be ideal spaces of scalar functions; $k_1, \dots, k_m$ nonnegative numbers. We shall write $X < (E_1^{k_1}, \dots, E_m^{k_m})$ if

$$\left\| |x_1|^{k_1} \cdots |x_m|^{k_m} \right\|_X \leqslant \|x_1\|_{E_1}^{k_1} \cdots \|x_m\|_{E_m}^{k_m} \quad (x_1 \in E_1, \dots, x_m \in E_m). \tag{6}$$

Below it is assumed that E and F are such dual spaces that $E \subseteq \mathcal{L}_2$, the superposition operator f acts from E to F, and the linear operator K acts from F to E.

2. An operator Q acting from one space into another will be called **asymptotically quadratic** if, for some quadratic operator Q_∞,

$$\lim_{\|x\|\to\infty} \frac{\|Qx - Q_\infty x\|}{\|x\|^2} = 0. \tag{7}$$

An important role in what follows is played by the asymptotic quadraticity of the superposition operator, acting from some ideal space $E \subseteq \mathcal{L}_2$ of functions with values in R^n into the space $\mathcal{L}_1$ of scalar functions,

$$Qx(s) = Q[s, x(s)], \tag{8}$$

where $Q(s,u)$ is some function satisfying the Carathéodory conditions. It can be shown that from the asymptotic quadraticity of Q it follows that the function $Q(s,u)$ admits the representation

$$Q(s,u) = \sum_{i,j=1}^{n} q_{ij}(s) u_i u_j + \omega(s,u), \tag{9}$$

where the function $\omega(s,u)/(u,u)$ tends in measure to zero as $u \to \infty$, and

$$Q_\infty x(s) = \sum_{i,j=1}^{n} q_{ij}(s) x_i(s) x_j(s) \quad (x = \{x_1, \dots, x_n\} \in E). \tag{10}$$

Lemma 1. Suppose the function $Q(s,u)$ admits the representation (9), where $q_{ij} \in E'_{ij}$ and $E_{ij} < (E_i, E_j)$. Suppose, moreover, that

$$\lim_{r\to\infty,\ \operatorname{mes} D\to 0} \frac{1}{r^2} \sup_{\|x\|_E = r} \int_D |Q[s, x(s)]|\, ds = 0. \tag{11}$$

Then Q is asymptotically quadratic.

A sufficient condition for (11) to be satisfied is the inequality

$$|Q(s,u)| \leq \sum_{i_1,\dots,i_n} a_{i_1\dots i_n}(s) |u_1|^{k_1} \cdots |u_n|^{k_n}, \tag{12}$$

where $k_1 + \dots + k_n \leq 2$, $a_{i_1 \dots i_n} \in (E'_{i_1 \dots i_n})^0$, and $E_{k_1 \dots k_n} < (E_1^{k_1}, \dots, E_n^{k_n})$.

Below, $Q(s,u)$ will always denote a function determining a bounded asymptotically quadratic operator Q acting from E into $\mathcal{L}_1$, $Q_\infty = 0$.

Let us introduce one more notation. Let $q(s) = (q_{ij}(s))$ be some symmetric matrix ($[q(s)]^* = q(s)$), and let $\bar{E} = \mathcal{L}_2$. Put

$$m(q; \bar{E}) = \sup_{x \neq 0} \langle qx, x \rangle / \|x\|^2_{\bar{E}}. \tag{13}$$

In practically important cases this quantity is easily computed or estimated.

3. Suppose that K is positive definite as an operator in $\mathcal{L}_2$. Denote by H the set of values of the operator $K^{1/2}$ on $\mathcal{L}_2$; it is well known that $H \subseteq E$. Below, $\bar{E}$ is some ideal space for which $E \subseteq \bar{E} \subseteq \mathcal{L}_2$. Using M. A. Krasnosel' skii' s fixed-point principle ([5,7]), we obtain:

Theorem 1. *Let the operator $K^{1/2} f K^{1/2}$ be completely continuous in $\mathcal{L}_2$, and let $h \in H$. Suppose that the inequality*

$$(u, f(s,u)) \leq \sum_{i,j=1}^{n} q_{ij}(s) u_i u_j + Q(s,u), \tag{14}$$

holds, where $q(s)$ is a symmetric matrix, and moreover $m(q; \bar{E}) \|K\|_{\bar{E}' \to E} < 1$.

Then equation (1) has a solution in H.

Suppose now that equation (1) is potential, i.e., that there exists a scalar function $\Phi(s,u)$ ($\Phi(s,0) = 0$) such that $\operatorname{grad} \Phi(s,u) = f(s,u)$. Then consider the functional defined on $\mathcal{L}_2$

$$\Phi(x) = \int_\Omega \Phi\big[s, K^{1/2} x(s)\big] \, ds. \tag{15}$$

Theorem 2. *Let the functional Φ be weakly lower semicontinuous, and let $h \in H$. Suppose that the inequality*

$$\Phi(s,u) \leq \frac{1}{2} \sum_{i,j=1}^{n} q_{ij}(s) u_i u_j + Q(s,u), \tag{16}$$

holds, where $q(s)$ is a symmetric matrix, and moreover $m(q; \bar{E}) \|K\|_{\bar{E}' \to \bar{E}} < 1$.

Then equation (1) has a solution in H.

We also give a uniqueness theorem.

Theorem 3. *Suppose that the inequality*

$$(u-v, f(s,u)-f(s,v)) < \sum_{i,j=1}^{n} q_{ij}(s)(u_i-v_i)(u_j-v_j) \quad (u \neq v), \tag{17}$$

holds, where $q(s)$ is a symmetric matrix, and moreover $m(q;\bar{E})\|K\|_{\bar{E}'\to E} \leq 1$.

Then equation (1) has at most one solution.

4. Suppose that the operator K has a finite number of negative eigenvalues, and that $(-\lambda_0)$ is the largest of them. Denote by $\widetilde{H}$ the set of values on $\mathcal{L}_2$ of the operator $\widetilde{K}^{1/2}$, where $\widetilde{K}$ is the positive definite self-adjoint quadratic root of the operator K^2. Note that the operator $\widetilde{K}^{1/2} f \widetilde{K}^{1/2}$ acts in $\mathcal{L}_2$, and, in the case where equation (1) is potential, the functional

$$\widetilde{\Phi}(x) = \int_\Omega \Phi\big[s, \widetilde{K}^{1/2}x(s)\big]\, ds. \tag{18}$$

Theorem 4. Let the operator $\widetilde{K}^{1/2} f \widetilde{K}^{1/2}$ be completely continuous and let $h \in \widetilde{H}$. Suppose the inequality

$$(u, f(s,u)) \leq \sum_{i,j=1}^{n} q_{ij} u_i u_j + Q(s,u), \tag{19}$$

holds, where q is a symmetric matrix, and $m(q;\mathcal{L}_2)\lambda_0 < -1$.

Then equation (1) has a solution in $\widetilde{H}$.

Theorem 5. Let the functional Φ be weakly upper semicontinuous and let $h \in \widetilde{H}$. Suppose the inequality

$$\Phi(s,u) \leq \frac{1}{2}\sum_{i,j=1}^{n} q_{ij} u_i u_j + Q(s,u), \tag{20}$$

holds, where q is a symmetric matrix and $m(q;\mathcal{L}_2)\lambda_0 < -1$.

Then equation (1) has a solution in $\widetilde{H}$.

Theorem 6. Suppose the inequality

$$(u-v, f(s,u)-f(s,v)) < \sum_{i,j=1}^{n} q_{ij}(u_i-v_i)(u_j-v_j) \quad (u \neq v), \tag{21}$$

holds, where q is a symmetric matrix, with $m(q;\mathcal{L}_2)\lambda_0 \leq -1$.

Then equation (1) has at most one solution.

5. Below, $C = K^{1/2} f K^{1/2}$ and $\Psi = \Phi$ in the case of positive definite K, and $C = \widetilde{K}^{1/2} f \widetilde{K}^{1/2}$ and $\Psi = \widetilde{\Phi}$ in the case when K has a finite number of negative eigenvalues. We give sufficient conditions for the complete continuity of C and the weak upper semicontinuity of Ψ.

Lemma 2. Suppose one of the following conditions is satisfied:

a) K is completely continuous as an operator from F into E;

b) K is regular as an operator from F into E, and f is an improving [(10)] operator from E into F;

c) $E = E_{u_0}$, $F = E'_{u_0}$.

Then C is completely continuous in $\mathcal{L}_2$, and, in the case when equation (1) is potential, Ψ is weakly upper semicontinuous on $\mathcal{L}_2$.

Lemma 3. Let equation (1) be potential, and suppose that the function $\Phi(s, u)$ satisfies the inequality

$$\Phi(s, u + h) - \Phi(s, u) - (f(s, u), h) \leq \mathcal{L}(s, u, h), \tag{22}$$

where $\mathcal{L}(s, u, h)$ $(\mathcal{L}(s, u, 0) \equiv 0)$ is a function satisfying the Carathéodory conditions and defining a superposition operator $L(x, h)$ acting from $E \times E$ into $\mathcal{L}_1$, improving for each fixed $x \in E$.

Then Ψ is weakly upper semicontinuous in $\mathcal{L}_2$.

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Note: Figure translations are in progress. See original paper for figures.