ON T-INVARIANCE OF THE COEFFICIENTS OF QUASILINEAR HYPERBOLIC EQUATIONS

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Abstract

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MATHEMATICS

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ON T-INVARIANCE OF THE COEFFICIENTS OF QUASILINEAR HYPERBOLIC EQUATIONS

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In § 1, necessary and sufficient conditions are established under which the operator of V. V. Nemytskii acts from a certain subset of the space C into the space L_q and is bounded. With the aid of this criterion, in § 2 the problem posed by S. L. Sobolev ([1]) of describing the set T of all functions satisfying the condition of T-invariance is solved. This problem arose in connection with the fact that membership in T of the coefficients of a quasilinear hyperbolic equation guarantees an a priori estimate of solutions in the norm W_2^l.

§ 1. Let F be a measurable bounded subset of the Euclidean space R_k; let G be a closed bounded subset of the Euclidean space R_m. Let the function $A(x, y)$ be defined on $F \times G$ and satisfy the Carathéodory condition, i.e., for each y it is measurable in x and for almost every x it is continuous in y.

Define on F the abstract function $\varphi(x)$ by the equality

$$\varphi(x) = A(x, y).$$

Lemma. $\varphi(x)$ is a measurable abstract function with values in the separable space $C(G)$.

Proof. Let $\Lambda_0 = \{y^{(n)}\}$ be a countable set everywhere dense in G. Since for any $\xi = \xi(y) \in C(G)$ the scalar functions $A(x, y^{(n)}) - \xi(y^{(n)})$ are measurable, the functions

$$\|\varphi(x) - \xi\|_C = \sup_{y^{(n)} \in \Lambda_0} \left|A(x, y^{(n)}) - \xi(y^{(n)})\right|$$

are also measurable. Hence, by ([4]), p. 87, the measurability of $\varphi(x)$ follows.

Let $G\{|y_i - y_i^0| \le a_i,\ i = 1, 2, \dots, m\}$. Denote by $D[F, G]$ (respectively by $M[F, G]$) the set of all continuous (respectively measurable) vector functions defined on F and taking values in G. Consider on $D[F,G]$ the V. V. Nemytskii operator

$$f(g) = A(x, g(x)).$$

Theorem 1. In order that the operator f act from $D[F,G]$ into $L_q(F)$ and be bounded on $D[F,G]$, it is necessary and sufficient that

$$\left\| \max_{y \in G} |A(x,y)| \right\|_{L_q(F)} < \infty.$$

Moreover,

$$\sup_{g \in D[F,G]} \|A(x, g(x))\|_{L_q(F)} = \left\| \max_{y \in G} |A(x,y)| \right\|_{L_q(F)}. \tag{1}$$

Proof. Sufficiency is obvious. We prove necessity. We shall show that

$$\sup_{g \in D[F,G]} \|A(x, g(x))\|_{L_q(F)} = \sup_{\mu \in M[F,G]} \|A(x, \mu(x))\|_{L_q(F)}. \tag{2}$$

For this it is enough to show that, for any function $\mu \in M[F,G]$,

$$\|A(x, \mu(x))\|_{L_q(F)} \leq \sup_{g \in D[F,G]} \|A(x, g(x))\|_{L_q(F)}. \tag{3}$$

Let $\mu \in M[F,G]$. Using N. Luzin' s theorem, construct a sequence of closed sets F_n such that on each set F_n the function $\mu(x)$ is continuous and $F_1 \subseteq F_2 \subseteq \cdots \subseteq F$, $\lim_{n\to\infty} m(F \setminus F_n) = 0$. Since for any $n = 1, 2, ...$*

$$\|A(x, \mu(x))\|_{L_q(F_n)} \leq \sup_{g \in D[F,G]} \|A(x, g(x))\|_{L_q(F_n)} \leq \sup_{g \in D[F,G]} \|\bar{A}(x, g(x))\|_{L_q(F)},$$

inequality (3) is satisfied. Thus equality (2) is proved.

We now consider the function $\varphi(x) = A(x,y)$. Since, by the lemma, the function $\varphi(x)$ is measurable, there is a sequence of measurable finite-valued functions

$$\varphi_s(x) = \sum_{i=1}^{N_s} \xi_i \chi_{E_i^{(s)}}(x), \qquad \xi_i = \xi_i(y) \in C(G),$$

converging to $\varphi(x)$ almost everywhere on F. But then ((1), pp. 295-296) there is a sequence of closed sets F_n, $F_1 \subseteq F_2 \subseteq \dots \subseteq F$, $\lim_{n\to\infty} m(F \setminus F_n) = 0$, such that on each set F_n, $\varphi_s(x) \to \varphi(x)$ uniformly. Put

$$A_s(x,y) = \varphi(x) = \sum_{i=1}^{N_s} \chi_{E_i^{(s)}}(x)\xi_i(y).$$

From the fact that $\varphi_s(x) \to \varphi(x)$ uniformly on F_n, $n = 1,2,\dots$, it follows that $A_s(x,y) \to A(x,y)$ uniformly on $F_n \times G$, $n = 1,2,\dots$. Consequently,

$$\lim_{s\to\infty} \left\| \max_{y\in G} |A_s(x,y)| \right\|_{L_q(F_n)} = \left\| \max_{y\in G} |A(x,y)| \right\|_{L_q(F_n)}, \qquad n = 1,2,\dots; \quad (4)$$

$$\lim_{s\to\infty} \sup_{\mu\in M[F,G]} \|A_s(x,\mu(x))\|_{L_q(F_n)} = \sup_{\mu\in M[F,G]} \|A(x,\mu(x))\|_{L_q(F_n)}, \qquad n = 1,2,\dots. \quad (5)$$

Taking (2), (4), (5) into account, we obtain

$$\left\| \max_{y\in G} |A(x,y)| \right\|_{L_q(F_n)} = \lim_{s\to\infty} \left\| \max_{y\in G} |A_s(x,y)| \right\|_{L_q(F_n)} =$$

$$= \lim_{s\to\infty} \sup_{\mu\in M[F,G]} \|A_s(x,\mu(x))\|_{L_q(F_n)} = \sup_{\mu\in M[F,G]} \|A(x,\mu(x))\|_{L_q(F_n)} \le$$

$$\le \sup_{g\in D[F,G]} \|A(x,g(x))\|_{L_q(F)}.$$

Thus,

$$\left\| \max_{y\in G} |A(x,y)| \right\|_{L_q(F_n)} \le \sup_{g\in D[F,G]} \|A(x,g(x))\|_{L_q(F)}, \qquad n = 1,2,\dots.$$

But $F_1 \subseteq F_2 \subseteq \dots \subseteq F$ and $\lim_{n\to\infty} m(F \setminus F_n) = 0$. Therefore,

$$A = \left\| \max_{y\in G} |A(x,y)| \right\|_{L_q(F)} \le \sup_{g\in D[F,G]} \|A(x,g(x))\|_{L_q(F)} = B.$$

The necessity is proved. Equality (1) follows from the last inequality ($A \le B$) and the obvious inequality $B \le A$.

§ 2. Let F be a bounded closed domain of the $(n+1)$-dimensional Euclidean space of variables (t,x), $x = (x_1,\dots,x_n)$; G a bounded–

* Note that, by virtue of (5), p. 374, there is a function $g \in D[F, G]$ such that $g(x) = \mu(x)$ for all x in F_n.

closed domain of m-dimensional Euclidean space defined by the system of inequalities

$$|y_i - y_i^0| \le a_i, \qquad i = 1, 2, \dots, m. \tag{6}$$

Let l be a natural number, $lp > n$, $1 \le p < \infty$. Consider on $F \times G$ a continuous function $A(t, x, y)$ satisfying two conditions:

1) for any $y \in G$, the function $A(t, x, y)$ has all generalized derivatives with respect to t, x up to order l inclusive, measurable in x for each fixed t;
2) let F_τ be the section of F by the hyperplane $t = \tau$, and suppose that for each τ the generalized derivatives

$$A_\alpha(\tau, x, y) = D_{t,x}^\alpha A(t, x, y), \qquad |\alpha| \le l,$$

belong to $C^l(G)$ for almost every $x \in F_\tau$, if $(l - |\alpha|)p \le n$, and for each $x \in F_\tau$, if $(l - |\alpha|)p > n$.

For brevity set

$$A_\alpha^\beta(t, x, y) = D_y^\beta D_{t,x}^\alpha A(t, x, y).$$

It is said ([2], p. 228) that the function $A(t, x, y)$ is T-invariant if* for any t and β, $|\beta| \le l$:

$$\alpha) \qquad \sup_{g \in D[F,G]} \left\| [A_\alpha^\beta(t, x, y)]_{y=g(t,x)} \right\|_{L_{\frac{1}{1/p - (l-|\alpha|)/n}}(F_t)} < \infty$$

when $(l - |\alpha|)p < n$;

$$\beta) \qquad \sup_{g \in D[F,G]} \left\| [A_\alpha^\beta(t, x, y)]_{y=g(t,x)} \right\|_{L_q(F_t)} < \infty$$

when $(l - |\alpha|)p = n$, where q is any number between 1 and ∞;

$$\gamma) \qquad \sup_{g \in D[F,G]} \left\| [A_\alpha^\beta(t, x, y)]_{y=g(t,x)} \right\|_{C(F_t)} < \infty$$

when $(l - |\alpha|)p > n$.

Theorem 2. Let the function $A(t, x, y)$ satisfy conditions 1), 2). For the T-invariance of the function $A(t, x, y)$ it is necessary and sufficient that for any t and β, $(|\beta| \le l)$:

$$\text{A.} \quad \left\| \max_{y \in G} |A_\alpha^\beta(t,x,y)| \right\|_{L_{\frac{1}{1/p-(l-|\alpha|)/n}}(F_t)} < \infty$$

when $(l - |\alpha|)p < n$;

$$\text{B.} \quad \left\| \max_{y \in G} |A_\alpha^\beta(t,x,y)| \right\|_{L_q(F_t)} < \infty$$

when $(l - |\alpha|)p = n$;

$$\text{C.} \quad A_\alpha^\beta(t,x,y) \in C(F_t \times G) \quad \text{when } (l - |\alpha|)p > n.$$

Proof. Let $t = \tau$ and $A_\tau(x,y) = A_\alpha^\beta(\tau,x,y)$. Put

$$s = \begin{cases} \dfrac{1}{1/p - (l-|\alpha|)/n}, & \text{if } (l-|\alpha|)p < n, \\ q, & \text{if } (l-|\alpha|)p = n. \end{cases}$$

By virtue of (1),

$$\sup_{h \in D[F_\tau, G]} \left\| [A_\tau(x,y)]_{y=h(x)} \right\|_{L_s(F_\tau)} = \left\| \max_{y \in G} |A_\tau(x,y)| \right\|_{L_s(F_\tau)}. \tag{7}$$

Since for any function $h(x) \in D[F_\tau, G]$ there is, by virtue of (5), p. 374, a function $g(t,x) \in D[F,G]$ such that $g(\tau,x) = h(x)$, we have

$$\sup_{h \in D[F_\tau, G]} \left\| [A_\tau(x,y)]_{y=h(x)} \right\|_{L_s(F_\tau)} = \sup_{g \in D[F,G]} \left\| [A_\alpha^\beta(\tau,x)]_{y=g(\tau,x)} \right\|_{L_s(F_\tau)}.$$

* Recall that $D[F,G]$ denotes the collection of all continuous vector functions mapping F into G.

Noting that $\max_{y \in G} x\, |A_\tau(x,y)| = \max_{y \in G} |A_\alpha^\beta(\tau,x,y)|$ and using (7), (8), we obtain

$$\sup_{g \in D[F,G]} \left\| [A_\alpha^\beta(\tau,x,y)]_{\cdot = g(\tau,x)} \right\|_{L_{sF_\tau}} = \left\| \max_{y \in G} |A_\alpha^\beta(\tau,x,y)| \right\|_{L_s(F_t)}.$$

Thus, condition α) is equivalent to condition A, condition β to condition B; the equivalence of conditions γ) and C) is obvious.

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[1] S. L. Sobolev, *Fund. Math.*, **47**, 3 (1959).
[2] S. L. Sobolev, Some applications of functional analysis in mathematical physics, L., 1950.
[3] S. L. Sobolev, *Sur les équations aux dérivées partielles hyperboliques non-linéaires*, Roma, 1961.
[4] E. Hille, R. Phillips, *Functional Analysis and Semi-Groups*, IL, 1962.
[5] I. P. Natanson, *Theory of Functions of a Real Variable*, M., 1957.

Note: Figure translations are in progress. See original paper for figures.