

# ON NONLINEAR PROBLEMS OF OCEANIC CIRCULATION

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## Abstract

## Full Text

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*GEOPHYSICS*

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# ON NONLINEAR PROBLEMS OF OCEANIC CIRCULATION

In recent years a number of investigations have been carried out on the theory and methods of solving nonlinear problems of the dynamics of the baroclinic ocean (<sup>1-4</sup>). In the present paper a further development of this theory is given on the basis of the method of splitting the complex operators of the problem into the simplest ones. The methodological basis for posing the problem is the construction of an equation for the pressure in the ocean, the principles of which have already been set forth in a linearized model (<sup>5</sup>). We shall write the equations of the dynamics of ocean currents in the form

$$\begin{aligned} \frac{du}{dt} - lv &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\partial}{\partial z} \nu \frac{\partial u}{\partial z} + \mu \Delta u, \\ \frac{dv}{dt} + lu &= -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\partial}{\partial z} \nu \frac{\partial v}{\partial z} + \mu \Delta v, \\ \frac{\partial p}{\partial z} &= g\rho, \end{aligned} \quad (1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

$$\frac{d\rho}{dt} + \Gamma w = \frac{\partial}{\partial z} \nu_1 \frac{\partial \rho}{\partial z} + \mu_1 \Delta \rho.$$

Here  $u, v, w$  are the components of the velocity vector;  $p$  and  $\rho$  are the deviations of the pressure and density of water from the standard values  $\bar{p}$  and  $\bar{\rho}$ ;  $\Gamma = d\bar{\rho}/dz$ ;  $l$  is the Coriolis parameter. Note that here the  $z$ -axis is directed from the ocean surface downward. Following (<sup>5</sup>), from the equation for the free surface  $\zeta$  we pass to the condition relating the functions  $p$  and  $w$ ,

$$\frac{\partial p}{\partial t} + \bar{g}\rho w = Dp_0 - \left( u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} \right) \quad \text{for } z = 0, \quad (2)$$

where

$$Dp_0 = \frac{\partial p_0}{\partial t} + u \frac{\partial p_0}{\partial x} + v \frac{\partial p_0}{\partial y};$$

$p_0$  is the atmospheric pressure at the level of the ocean surface. To system (1) we adjoin the conditions

$$\nu \frac{\partial u}{\partial z} = -\frac{\tau_{xz}}{\rho}, \quad \nu \frac{\partial v}{\partial z} = -\frac{\tau_{yz}}{\rho}, \quad \frac{\partial \rho}{\partial z} = \gamma \quad \text{for } z = 0, \quad (3)$$

where  $\tau_{xz}, \tau_{yz}$  are the prescribed wind stresses on the ocean surface;  $\nu\gamma$  is the density flux due to changes in the temperature and salinity of the surface layer of water—a prescribed function of the coordinates and time. At the ocean bottom the conditions are imposed

$$u = 0, \quad v = 0, \quad w = u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y}, \quad \frac{\partial \rho}{\partial z} = 0 \quad \text{for } z = H, \quad (4)$$

where  $z = h(x, y)$  is the equation of the ocean-bottom surface, and  $H$  is a certain “mean” depth of the ocean. It is assumed here that the orography of the ocean can be taken into account within the framework of the theory of small perturbations. The question of allowing for orography is discussed more fully in <sup>(1)</sup>.

Let us suppose that the ocean is bounded by a cylindrical surface  $S$ , on which we impose the no-slip condition and the equality to zero of the flux of density through the boundary  $S$

$$u = 0, \quad v = 0, \quad \partial \rho / \partial n = 0 \quad \text{on } S, \quad (5)$$

where  $\mathbf{n}$  is the normal to the surface  $S$ .

As initial data we take

$$u = u^0, \quad v = v^0, \quad \rho = \rho^0 \quad \text{for } t = 0. \quad (6)$$

We shall seek the solution of problem (1)–(6) by means of a difference method. To this end, the entire interval  $0 \leq t < T$  is divided into partial intervals of width  $\Delta t = \tau$ , and within each such interval  $t_j \leq t \leq t_{j+1}$  the problem under consideration is approximately represented in the form of two problems. First we solve the problem of transport of the substances  $\bar{\rho}u$ ,  $\bar{\rho}v$ , and  $\rho$  along particle trajectories

$$du/dt = 0, \quad dv/dt = 0, \quad d\rho/dt = 0 \quad (7)$$

under the condition that the functions  $u$ ,  $v$ , and  $\rho$  are prescribed at those boundary points where the velocity vector is directed into the domain  $D$  in which the solution is defined. As initial data we take the functions

$$u = u^j, \quad v = v^j, \quad \rho = \rho^j \quad \text{for } t = t_j. \quad (8)$$

Suppose that the solution of problem (7)–(8) at the time  $t_{j+1}$  has been found and denote it by the index  $j + 1/3$ . Then we shall have  $u^{j+1/3}$ ,  $v^{j+1/3}$ , and  $\rho^{j+1/3}$ . Next we choose the obtained solutions as initial data for the solution of the problem of nonstationary turbulent diffusion on the interval  $t_j \leq t \leq t_{j+1}$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial z} \nu \frac{\partial u}{\partial z} + \mu \Delta u, \quad \frac{\partial v}{\partial t} = \frac{\partial}{\partial z} \nu \frac{\partial v}{\partial z} + \mu \Delta v, \quad \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial z} \nu_1 \frac{\partial \rho}{\partial z} + \mu_1 \Delta \rho \quad (9)$$

with conditions on the ocean surface

$$\nu \partial u / \partial z = -\tau_{xz} / \bar{\rho}, \quad \nu \partial v / \partial z = -\tau_{yz} / \bar{\rho}, \quad \partial \rho / \partial z = \gamma \quad \text{for } z = 0 \quad (10)$$

and at the bottom

$$u = 0, \quad v = 0, \quad \partial \rho / \partial z = 0 \quad \text{for } z = H. \quad (11)$$

On the coastal surface  $S$  we impose the conditions

$$u = 0, \quad v = 0, \quad \partial \rho / \partial n = 0 \quad \text{on } S. \quad (12)$$

As “initial” data for  $t = t_j$  we take the following:

$$u = u^{j+1/3}, \quad v = v^{j+1/3}, \quad \rho = \rho^{j+1/3} \quad \text{for } t = t_j. \quad (13)$$

Here it is assumed that

$$d/dt = \partial/\partial t + u^j \partial/\partial x + v^j \partial/\partial y + w^j \partial/\partial z,$$

and  $u^j$ ,  $v^j$ ,  $w^j$  are known functions of the coordinates.

After problem (7)–(8) has been solved and the hydrological characteristics  $u$ ,  $v$ , and  $\rho$  at the time  $t_{j+1}$  have been obtained, which we denote by  $u^{j+2/3}$ ,  $v^{j+2/3}$ ,  $\rho^{j+2/3}$ , we take these values now as “initial” ones for  $t = t_j$  in order to solve

the problem of adapting the hydrological fields. As a result we arrive at a new problem on the same time interval  $t_j \leq t \leq t_{j+1}$

$$\begin{aligned} \frac{\partial u}{\partial t} - lw &= -\frac{1}{\bar{\rho}} \frac{\partial p}{\partial x}, & \frac{\partial v}{\partial t} + lu &= -\frac{1}{\bar{\rho}} \frac{\partial p}{\partial y}, \\ \partial p / \partial z &= g\rho, \end{aligned} \quad (14)$$

$$\partial u / \partial x + \partial v / \partial y + \partial w / \partial z = 0,$$

$$\partial \rho / \partial t + \Gamma w = 0$$

with the conditions

$$\begin{aligned} \frac{\partial \rho}{\partial t} + g\bar{\rho}w &= Dp_0 - \left( u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} \right) \quad \text{for } z = 0, \\ w &= u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} \quad \text{for } z = H \end{aligned} \quad (15)$$

and on the surface  $S$

$$\mathbf{u} \cdot \mathbf{n} = 0. \quad (16)$$

The initial data have the form

$$u = u^{j+2/3}, \quad v = v^{j+2/3}, \quad \rho = \rho^{j+2/3}. \quad (17)$$

In the work <sup>(6)</sup>, in a somewhat simpler formulation, a theorem was proved on the convergence of the solutions of problems (7)–(8), (9)–(13), and (14)–(17) to the exact solution of problem (1)–(6). The well-posedness of problem (7)–(8) and (9)–(13) has been well studied, and the well-posedness of problem (14)–(17) was established in the work <sup>(5)</sup>. Equations (7) and (9) have been well studied, and solutions of these equations under the corresponding boundary conditions and initial data are found by means of finite-difference methods that have been considered in the literature with sufficient completeness.

We proceed to the consideration of the problem of dynamic adjustment of the fields (14)–(17). We write the system of equations (12) in difference form, using an implicit approximation in  $t$ . Then we shall have

$$\frac{u - u^{j+1/3}}{\tau} - lw = -\frac{1}{\bar{\rho}} \frac{\partial p}{\partial x},$$

$$\begin{aligned} \frac{v - v^{j+1/3}}{\tau} + lu &= -\frac{1}{\bar{\rho}} \frac{\partial p}{\partial y}, \\ \frac{\partial p}{\partial z} &= g\rho, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0, \\ \frac{\rho - \rho^{j+1/3}}{\tau} + \Gamma w &= 0. \end{aligned} \quad (18)$$

Here the index  $j + 1$  at the unknowns has been omitted for simplicity.

From the first two equations of system (18) we express  $u$  and  $v$  in terms of  $p$ . Then we obtain

$$\begin{aligned} u &= \frac{1}{1 + \alpha^2} \left[ u^{j+1/3} + \alpha v^{j+1/3} - \frac{\tau}{\bar{\rho}} (p_x + \alpha p_y) \right], \\ v &= \frac{1}{1 + \alpha^2} \left[ v^{j+1/3} - \alpha u^{j+1/3} - \frac{\tau}{\bar{\rho}} (p_y - \alpha p_x) \right], \end{aligned} \quad (19)$$

where  $\alpha = \tau l$ .

From the last three equations of system (18) we find

$$\frac{1}{g\tau} \frac{\partial}{\partial z} \frac{1}{\Gamma} \frac{\partial p}{\partial z} = \frac{1}{\tau} \frac{\partial}{\partial z} \frac{\rho^{j+1/3}}{\Gamma} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}. \quad (20)$$

In equation (20) we substitute  $u$  and  $v$  from (19) and take into account the linear dependence of the parameter  $l$  on  $y$ . Then we arrive at an equation for  $p$

$$\frac{\partial}{\partial z} \frac{1}{\chi} \frac{\partial p}{\partial z} + m_0^2 \left( \Delta p + \beta \tau \frac{\partial p}{\partial x} \right) = f, \quad (21)$$

where

$$f = \frac{m_0^2}{\tau} \left[ (u^{j+1/2} + \alpha v^{j+1/2})_x + (v^{j+1/2} - \alpha u^{j+1/2})_y \right] + g \frac{\partial}{\partial z} \left( \frac{\rho^{j+1/2}}{\chi} \right), \quad (22)$$

$$m_0^2 = \frac{g\Gamma_0}{\rho} \frac{\tau^2}{1 + \tau^2 l^2}, \quad \Gamma = \Gamma_0 \chi(z).$$

The boundary conditions for equation (21) will be the following:

$$\frac{1}{\Gamma} \frac{\partial p}{\partial z} - \frac{1}{\rho} p = \delta^j, \quad \text{for } z = 0,$$

$$\frac{\partial p}{\partial z} = g\rho^j - g\Gamma \left( u^j \frac{\partial h}{\partial x} + v^j \frac{\partial h}{\partial y} \right), \quad \text{for } z = H, \quad (23)$$

where

$$\delta^j = \frac{g}{\Gamma} \rho^j - \frac{1}{\rho} \left[ Dp_0 - \left( u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} \right) \right]^j. \quad (24)$$

On the surface  $S$  we impose the condition

$$\frac{1}{\alpha} \frac{\partial p}{\partial n} - \frac{\partial p}{\partial s} = -\frac{\rho}{\tau} \left( \frac{1}{\alpha} \mathbf{u}^{j+1/3} \cdot \mathbf{n} - \mathbf{u}^{j+1/3} \cdot \mathbf{s} \right). \quad (25)$$

After the solution of equation (21) with the boundary conditions (23), (25) has been found, all the necessary hydrological quantities are found by means of the equations of system (18) with the pressure already known. The characteristics obtained at the last stage are interpreted as an approximate solution of the problem at the time  $t_{j+1}$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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