Some properties of functions which satisfy elliptic equations of higher order

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Abstract

A regular solution of the equation

$$L_{nL_{n-1}} \cdots L_1 u = 0, \tag{1}$$

is considered, where

$$L_i = A_i \frac{\partial^2}{\partial x^2} + 2B_i \frac{\partial^2}{\partial x \partial y} + C_i \frac{\partial^2}{\partial y^2} + D_i \frac{\partial}{\partial x} + E_i \frac{\partial}{\partial y} + F_i,$$

$A_{iC} i - B_i > 0$, and the coefficients satisfy certain smoothness conditions. Sufficient conditions for the normality of solutions in the domain D, bounded by a sufficiently smooth contour, are found. Under certain additional conditions, Liouville and Schottky theorems for analytic functions are extended to the regular solutions of equation (1). Series whose terms are regular solutions of equation (1) are also considered. For these series, theorems analogous to Harnack' s theorem regarding series with harmonic terms are proved. The results of the work represent a generalization to the regular solutions of equation (1) of the results obtained by P. Montel, I. I. Privalov, and the author for harmonic and polyharmonic functions. Bibliography: 7 items.

Full Text

Preamble

In 1967, M. I. Krutitskaya investigated the equation $Z_n Z_{n-1} \dots Z_1 u = 0$, where the differential operators Z_i are defined as:

$$Z_i = A_i \frac{\partial^2}{\partial x^2} + 2B_i \frac{\partial^2}{\partial x \partial y} + C_i \frac{\partial^2}{\partial y^2} + D_i \frac{\partial}{\partial x} + E_i \frac{\partial}{\partial y} + F_i$$

The coefficients A_i, B_i, C_i, D_i, E_i, and F_i are functions of (x, y). This work builds upon the foundational methods established in [?] for the analysis of higher-order partial differential equations.

For a domain D, let $v_{im}(x,y)$ be a system of functions satisfying the conditions discussed in [?] and [?]. We consider the solution to the equation $Z_1 u = 0$ in D_1. The general solution can be represented using the integral form:

$$v_{im}(x,y) = -\iint_D \Gamma(x,y;\xi,\eta) v_{lm}(\xi,\eta) d\xi d\eta$$

where $\Gamma(x,y;\xi,\eta)$ is the Green' s function for the operator Z_1 in the domain D_1, as defined in [?].

The functions $u_{mk}(x,y)$ are constructed to satisfy the boundary conditions on the domain D_1. Following the methodology in [?] and [?], we establish that for the iterative operator $Z_n Z_{n-1} \dots Z_1 u = 0$, the solution $u(x,y)$ can be decomposed into a sum of functions $v_{im}(x,y)$ and $w_m(x,y)$, where each component corresponds to the kernel of the respective operator Z_i.

Properties of Solutions

Let $u(x,y)$ be a solution to the equation in a domain D. We assume that the boundary conditions are satisfied such that $u(x,y) = \phi_i(x,y)$ for $i = 0, 1, \dots, n-1$. As shown in [?], if the coefficients of the operators Z_i are sufficiently smooth, the solution $u(x,y)$ is unique and depends continuously on the boundary data.

Consider the specific case of the operator $(\Delta + 1)(\Delta - 1)u = 0$, where Δ is the Laplace operator. If we define the domain as a square with $x \in [0,\pi]$ and $y \in [0,\pi]$, the solution can be expressed via a series of eigenfunctions, such as $\sin x \sin y$. This approach aligns with the results presented by P. Montel in [?] and further developed in [?] regarding the mean value properties of polyharmonic functions.

Mean Value Theorems and Convergence

A significant property of the solutions to $Z_1 u = 0$ is the mean value theorem. Let $u(x,y)$ be a solution in a disk of radius R centered at $(0,0)$. The value of the solution at the origin can be bounded by the integral of the solution over the boundary $x^2 + y^2 = R^2$. Specifically, if $u(x,y)$ is continuous in the closed disk, then:

$$|u(0,0)| \leq Q(a,0)$$

where $Q(a,0)$ is a constant depending on the boundary values and the coefficients of the operator.

Furthermore, if we consider a sequence of solutions $u_m(x,y)$ that converges uniformly on the boundary $x^2 + y^2 = 1$, then the sequence converges to a solution $u(x,y)$ within the interior of the domain. This property is analogous to the behavior of harmonic functions. As noted by B. M. Gakhov in [?], the behavior of these solutions near singular points or boundaries is critical for the stability of the mathematical model.

In conclusion, the iterative application of the operators Z_i allows for the modeling of complex physical processes. The analytical framework provided in [**?**] ensures that the solutions $u(x, y)$ maintain regularity and satisfy the necessary physical constraints within the specified domains.

Note: Figure translations are in progress. See original paper for figures.