ON THE PROPERTIES OF THE DECOMPOSITION INTO BUNDLES OF THE CAUSAL \(S\)-MATRIX

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Abstract

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ON THE PROPERTIES OF THE DECOMPOSITION INTO BUNDLES OF THE CAUSAL S-MATRIX

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In the present paper we shall prove space-like properties of the decomposition into bundles of vacuum mean values of radiation operators, as well as of matrix elements of the S-matrix within the framework of N. N. Bogolyubov's axiomatic causal S-matrix theory ([1]). The properties of the decomposition into bundles of the S-matrix have recently been formulated ([2]) and proved ([3]) within the framework of Wightman' s axiomatic quantum field theory ([4]). For simplicity we shall consider the self-action of one real scalar field $\varphi(x)$.

Consider

$$S = \sum_{N=0}^{\infty} \frac{(-i)^N}{N} \int (d^4x)_N \, h_N(x)_N : \varphi(x_1) \dots \varphi(x_N) :, \tag{1}$$

where the coefficient functions $h_N(x)_N$ are C-numbers with the property $h_N(x_{\alpha N}) = h_N(x_N)$. In formula (1) the following notation has been introduced:

$$(z)_N = (z_1, \dots, z_N), \qquad (z_\alpha)_N = (z_{\alpha_1}, \dots, z_{\alpha_N}),$$

$$\int (d^4x)_N = \int \cdots \int d^4x_1 \dots dx_N^4.$$

Following N. N. Bogolyubov ([1]), we define the causality condition according to

$$\iint d^4x \, d^4y f(x) g(y) \frac{\delta}{\delta\varphi(x)} \left(\frac{\delta S}{\delta\varphi(y)} S^+ \right) = 0 \tag{2}$$

for arbitrary $f(x)$, $g(y) \in S(R^4)$, satisfying the condition $f(x)g(y) = 0$ for time-like intervals $(x-y)^2 \geq 0$ and $(x^0 - y^0) \geq 0$.

Let us introduce into consideration the radiation-operator-valued generalized function

$$H_N(f) = \int (d^4x)_N f(x)_N \frac{\delta^N S}{\delta\varphi(x_1)\dots\delta\varphi(x_N)} S^+ =$$

$$= (-i)^N \int (d^4x)_N \theta(x_1^0 - x_2^0)\dots\theta(x_{N-1}^0 - x_N^0) J(x_1)\dots J(x_N) \sum_{p(1,\dots,N)} f(x_\alpha)_N, \quad (3)$$

defined in $D_S \subset H$ for any $f(x)_N \in S_0(R^{4N}) \subset S(R^{4N})$, vanishing together with its derivatives of sufficiently high order when $x_1 = x_2 = \dots = x_N$. The summation in (2) is taken over all possible permutations $\alpha_1, \dots, \alpha_N$ of the numbers $1, 2, \dots$. By $J(x) = i\,\delta S/\delta\varphi(x)$ we denote the operator of the bosonic current. We shall assume that

$$h_N(f) = i^N \langle 0|H_N(f)|0\rangle \in S_0'(R^{4N}) \quad (4)$$

for every $f(x_1, \dots, x_N) \in S_0(R^{4N})$.

Theorem 1. Let a be an arbitrary space-like vector, $\lambda < 0$. If condition (2) is satisfied, the relation

$$\lim_{\lambda\to\infty} h_{m+n+r+s}(F_{\lambda a}^{(m+n+r+s)}) = h_{m+n}(f^{(m+n)})\, h_{r+s}(g^{(r+s)}) \quad (5)$$

holds for any

$$F_{\lambda a}^{(m+n+r+s)} = F^{(m+n+r+s)}(x,\, y - \lambda a)_{m+n,r+s} =$$

$$= f^{(m+n)}(x)_{m+n} g^{(r+s)}(y - \lambda a)_{r+s} \in S_0(R^{4(m+n+r+s)}), \quad (6)$$

where $f^{(m+n)}(x)_{m+n},\ g^{(r+s)}(y)_{r+s} \in S_0(R^{4(m+n)}, R^{4(r+s)})$.

One can indicate a relation analogous to (5) for matrix elements of the S-matrix.

Between the matrix elements $S_{m+r,n+s}(\mathbf{q}',\, \mathbf{p}';\, \mathbf{q},\, \mathbf{p})_{m,r;n,s} \in S'(R^{3(m+n+r+s)})$ and the vacuum averages of radiation operators $h_{m+n+r+s}(x',\, y';\, x,\, y)_{m,r;n,s} \in S'(R^{4(m+n+r+s)})$ there holds the relation

$$S_{m+r,n+s}\left(\exp\left[-i\lambda\left(\sum_{j=1}^{r} a\cdot p_j - \sum_{k=1}^{s} a\cdot p_{k+r}\right)\right]\tilde{f}^{(m+n)}\tilde{g}^{(r+s)}\right) = h_{m+n+r+s}(f^{(m+n)} g_{\lambda a}^{(r+s)}), \quad (7)$$

where

$$f^{(m+n)}(x)_{m+n} = \frac{1}{2\pi^{3(m+n)/2}} \int \frac{\exp\left[i\left(\sum_{l=1}^{m} q_l x_l - \sum_{t=1}^{n} q_{m+t} x_{m+t}\right)\right]}{\sqrt{2q_1^0 \dots 2q_{m+n}^0}} \tilde{f}^{(m+n)}(\mathbf{q})_{m+n}\,(dq)_{m+n},$$

$$q_i^0 = \sqrt{\mathbf{q}_i^2 + m^2}, \qquad i = 1, \dots, m+n, \tag{8}$$

where $\tilde{f}^{(m+n)}(\mathbf{q})_{m+n} \in S(\tilde{R}^{3(m+n)})$ and $f^{(m+n)}(R^{4(m+n)})$,

The function $g^{(r+s)}(y)_{r+s} \in S(R^{3(r+s)})$ has the same structure as $f^{(m+n)}(x)_{m+n}$, and $g_{\lambda a}^{(r+s)} = g^{(r+s)}(y - \lambda a)_{r+s}$, where a is an arbitrary space-like vector, $\lambda > 0$.

Theorem 2. Suppose the condition of Theorem 1 is fulfilled. Then

$$\lim_{\lambda \to \infty} S_{m+r,n+s}\left(\exp\left[-i\lambda\left(\sum_{j=1}^{r} a \cdot p_j - \sum_{k=1}^{s} a \cdot p_{r+k}\right)\right] \tilde{f}^{(m+n)} \tilde{g}^{(r+s)}\right) =$$

$$= S_{m+n}(\tilde{f}^{(m+n)})\, S_{r+s}(\tilde{g}^{(r+s)}) \tag{9}$$

for any $\tilde{f}^{(m+n)}(\mathbf{q}_{m+n}) \in S_0(\tilde{R}^{3(m+n)})$ and $\tilde{g}^{(r+s)}(\mathbf{p}_{r+s}) \in S_0(\tilde{R}^{3(r+s)})$.

It is obvious that the proof of Theorem 2 follows directly from Theorem 1. We divide the proof of the theorem into the proofs of several lemmas.

Lemma 1. Suppose the causality condition (1) is fulfilled. Then

$$H_{m+n+r+s}(F^{(m+n+r+s)}) = H_{m+n}(f^{(m+n)})\, H_{r+s}(g^{(r+s)}) \tag{10}$$

for any function

$$F^{(m+n+r+s)}(x, y)_{m+n,r+s} \in S_0(R^{4(m+n+r+s)}),$$

representable in the form

$$F^{(m+n+r+s)}(x, y)_{n+m,r+s} = f^{(m+n)}(x)_{m+n} g^{(r+s)}(y)_{r+s}, \tag{11}$$

provided $\{x\}_{n+m} \geq \{y\}_{r+s}$, where $f^{(m+n)}(x)_{m+n} \in S_0(R^{4(m+n)})$ and $g^{(r+s)}(y)_{r+s} \in S_0(R^{4(r+s)})$.

The proof of the lemma is based on the results of work (5).

Now, using this lemma and the completeness condition for the system of eigen-amplitudes of the 4-energy-momentum operator $\hat{P}^{\mu}$, we obtain

$$h_{m+n+r+s}(F_a^{(m+n+r+s)}) = h_{m+n}(f^{(m+n)})\,h_{r+s}(g^{(r+s)}) + h^T_{m+n+r+s}(F_{\lambda a}^{(m+n+r+s)}), \tag{12}$$

where $h^T_{m+n+r+s}(F_{\lambda a}^{(m+n+r+s)})$, the truncated vacuum mean value of the radiation operators, is defined according to

$$\begin{aligned} h^T_{m+n+r+s}(F_{\lambda a}^{(m+n+r+s)}) = &\iint (d^4x)_{m+n-1}(d^4y)_{r+s}\theta(x_1^0 - x_2^0)\cdots \\ &\cdots\theta(x^0_{m+n-1} - x^0_{m+n})\theta(y_1^0 - y_2^0)\cdots\theta(y^0_{r+s-1} - y_{r+s}) \\ &\times\left[\sum_{\nu=1}^{\infty}\frac{1}{\nu!}\int (dk)_\nu\langle 0|J(x_1)\cdots J(x_{m+n})|(k)_\nu\rangle\langle(k)_\nu|J(y_1)\cdots J(y_{r+s})|0\rangle\right] \\ &\times\sum_{P(1,\dots,m+n)} f^{(m+n)}(x_\alpha)_{m+n}\sum_{P(1,\dots,r+s)} g^{(r+s)}(y_\beta - \lambda a)_{r+s} \end{aligned} \tag{13}$$

for arbitrary $f^{(m+n)}(x)_{m+n} \in S_0(R^{4(m+n)})$ and $g^{(r+s)}(y)_{r+s} \in S_0(R^{4(r+s)})$.

Lemma 2. The Fourier transform of $h^T_{m+n+r+s}(F_{\lambda a}^{(m+n+r+s)})$ belongs to the space $S_0(\widetilde{R}^{3(m+n+r+s)})$.

Indeed, using the completeness properties of the amplitude of the 4-energy-momentum operator $\hat{P}^{\mu}$, from (11) we obtain

$$\begin{aligned} h^T_{m+n+r+s}(F_{\lambda a}^{(m+n+r+s)}) = i^{m+n+r+s}&\int\cdots\int I(\dots,\mu^2_{\alpha 3},\dots;\dots,m^2_{ij},\dots;M^2) \\ &\times\widetilde{D}^{(-)}_{0,\mu^2_{\alpha 3},m^2_{ij},M^2}(e^{i\lambda ak}\tilde{f}_2^{(m+n)}\tilde{g}_2^{(r+s)}) \\ &\times\prod_{\alpha\le\beta\le 1}^{m+n+1} d\mu^2_{\alpha\beta}\prod_{i<j\le 1}^{r+s+1} dm^2_{ij}\,dM^2, \end{aligned} \tag{14}$$

where

$$\widetilde{D}^{(-)}_{0,\mu^2_{\alpha 3},m^2_{ij},M^2} \in S'(R(\mu^2_{\alpha 3}, m^2_{ij}, M^2))$$

and is a Lorentz-invariant generalized function. Since the carrier of the generalized function $I(\dots,\mu^2_{\alpha\beta},\dots;\dots,m^2_{ij},\dots;M^2) \in S'(R)$ lies in the region

$$\{\dots,\mu^2_{\alpha\beta},\dots;\dots,m^2_{ij},\dots;M^2\} > 0, \tag{15}$$

the integral (14) exists, and therefore $h^T_{m+n+r+s}(F^{(m+n+r+s)}_{\lambda a})$ depends linearly and continuously on the functions $\tilde{f}_2(q,k)_{m+n-1,0} \in S_0(\widetilde{R}^{3(m+n)})$ and $\tilde{g}_2^{(r+s)}(p,k)_{r+s-1,0} \in S_0(\widetilde{R}^{3(r+s)})$.

Lemma 3. Let a be an arbitrary spacelike vector, $\lambda > 0$. Then

$$\lim_{\lambda\to\infty} \lambda^N h^T_{m+n+r+s}(F^{(m+n+r+s)}_{\lambda a}) = 0 \tag{16}$$

for any $F^{(m+n+r+s)} \in S_0(R^{4(m+n+r+s)})$ of the form (5).

We note that the proof of this lemma reduces, on the basis of formula (13), to proving the relation

$$\lim_{\lambda\to\infty} \lambda^N \widetilde{D}^{(-)}_{0,\mu^2_{\alpha\beta},m^2_{ij},M^2}(e^{i\lambda a k'} \tilde{f}_2^{(m+n)} \tilde{g}_2^{(r+s)}) = 0 \tag{17}$$

for $\tilde{f}_2^{(m+n)}(q,k)_{m+n-1,0} \in S_0(\widetilde{R}^{3(m+n)})$ and $\tilde{g}_2^{(r+s)}(p,k)_{r+s-1} \in S_0(\widetilde{R}^{3(r+s)})$.

In (16) the reference frame $d \equiv (0,a,0,0)$ is chosen, and k' is the component of the vector $k = (k^0,k^1,k^2,k^3)$. Araki [6] proved that relation (17) holds in the case when $\widetilde{D}^{(-)}_{0,\mu^2_{\alpha\beta},m^2_{ij},M^2}$ is a Lorentz-invariant generalized function. Since in the present case $\widetilde{D}^{(-)}_{0,\mu^2_{\alpha 3},m^2_{ij},M^2}$ is a Lorentz-invariant generalized function, Lemma 3 is thereby proved.

On the basis of the lemmas, Theorem 1 is completely proved and, consequently, Theorem 2 as well.

Now define the extension of the radiation operator $H_N(f)$, given by formula (3) in $S_0(R^{4N})$, to the whole space $S(R^{4N})$ according to

$$H^C_N(f) = \int (d^4x)_N\, \theta(x^0_1 - x^0_2) \cdots \theta(x^0_{N-1} - x^0_N) J(x_1) \cdots J(x_N) \times$$

$$\times \sum_{P(1,\dots,N)} f(x_\alpha)_N + \int (d^4x)_N f(x)_N C^\Lambda_N(x)_N \tag{18}$$

for any function $f(x)_N \in S(R^{4N})$, where $C^\Lambda_N(x)_N$ is expressed as a sum of symmetrized T-products of chains of quasilocal operators $\Lambda_n(x_1,\dots,x_n)$ (5), possessing the properties of locality, Hermiticity, symmetry, and local commutativity.

Since

$$h^C_N(f) = i^N \langle 0|H^C_N(f)|0\rangle \in S'(R^{4N}) \tag{19}$$

for any $f \in S(R^{4N})$, the properties of the cluster decomposition of the vacuum mean values of the radiation operators with coinciding arguments will be defined according to

$$\lim_{\lambda\to\infty} h^C_{m+n+r+s}(F_{\lambda a}) = h^C_{m+n}(f) h^C_{r+s}(g) \tag{20}$$

for any function $F(x,y)_{m+n,r+s} = f(x)_{m+n} g(y)_{r+s} \in S(R^{4(m+n+r+s)})$, where $f(x)_{m+n} \in S(R^{4(m+n)})$ and $g(y)_{r+s} \in S(R^{4(r+s)})$, with causally independent carriers $\{x\}_{m+n} > \{y\}_{r+s}$.

Extending relation (7) to the whole space $S(R^{4(m+n+r+s)})$ and using formula (20), we obtain the cluster-decomposition properties of the elements of the S-matrix, expressed in the form of formula (9).

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Note: Figure translations are in progress. See original paper for figures.