ON A NONELLIPTIC DIRICHLET PROBLEM

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Abstract

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MATHEMATICS

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ON A NONELLIPTIC DIRICHLET PROBLEM

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Consider an elliptic system of two differential equations with real coefficients, written in complex form as a single equation

$$w_{\bar z\bar z} - A_1\bar w_z - A_2 w_z - A_3\bar w_{\bar z} - A_4 w_{\bar z} - A_5\bar w - A_6 w = F, \tag{1}$$

where $w = u_1 + iu_2$; $\bar w = u_1 - iu_2$; $\partial/\partial z = \frac{1}{2}(\partial/\partial x - i\partial/\partial y)$, $\partial/\partial\bar z = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$; $F(z,\bar z)$, $A_j(z,\bar z)$ $(j = 1,\dots,6)$ are analytic functions of the variables z and $\bar z$ in some cylindrical domain (T, T^*), T being some domain of the plane $z = x + iy$, and T^* the reflection of the domain T with respect to the real axis.

Every twice continuously differentiable solution of equation (1) in the domain T is an analytic function ([1,2]).

For equation (1) the first boundary-value problem is studied in the following formulation:

Find a solution of equation (1) in a bounded simply connected domain $D \subset T$ with boundary $L \in A^{(2,\alpha)}$, belonging to the class $C^2(D) \cap C^{(1,\alpha)}(D + L)$, under the condition

$$w|_L = f, \tag{2}$$

where f is a given complex function of class $C^{(k,\alpha)}(a)$, and $k \geq 2$.

In the absence of lower-order terms ($A_j = 0,\ j = 1,\dots,6$), problem (2)–(1) is not Noetherian.* For example, in the disk $|z - z_0| \leq R$ the homogeneous problem has an infinite number of linearly independent solutions ([3]). However, problem (1)–(2) becomes Noetherian when terms containing $\bar w$ are added.

Theorem 1. *The Dirichlet problem for the equation*

$$\partial^2 w/\partial\bar z^2 - A_1(z,\bar z)\,\partial\bar w/\partial z = 0 \tag{3}$$

is Noetherian if

$$A_1(z,\bar z)\neq 0 \quad \text{on } L,$$

and the index of the problem is equal to

$$\frac{1}{\pi}\,[\arg A_1(z,\bar z)]_L-2,$$

where $[\ ]_L$ *denotes the increment of the argument of the function* $A_1(z,\bar z)$ *when the domain* D *is traversed once in the positive direction.*

Proof. Equation (3) is equivalent to the system

$$\partial w/\partial\bar z-u=0,$$

$$\partial u/\partial\bar z-A_1\bar u=0, \tag{4}$$

* Here problem (1)–(2) is called Noetherian if the homogeneous problem has a finite number of linearly independent solutions, and the nonhomogeneous problem is solvable subject to a finite number of orthogonality conditions on the function f.

which is equivalent to the system of Volterra integral equations

$$w-\int_0^{\bar z}u(z,\tau)\,d\tau=\psi(z),\qquad u-\int_0^{\bar z}A_1(z,\tau)\,\overline{u(\bar z,\bar\tau)}\,d\tau=\varphi''(z), \tag{5}$$

where $\psi(z)$ and $\varphi''(z)$ are arbitrary functions holomorphic in the domain D, with $\varphi(0)=\varphi'(0)=0$.

Solving the system (5), we obtain a general representation of an arbitrary solution, regular in the domain D, of equation (3) in terms of two arbitrary holomorphic functions $\varphi(z)$ and $\psi(z)$:

$$\begin{aligned} w=\psi+z\varphi''+A_0\bar\varphi+B\varphi'+C\varphi+\\ +\int_0^z F_1(z,\bar z,t)\varphi(t)\,dt+\int_0^{\bar z}F_2(z,\bar z,\tau)\overline{\varphi(\bar\tau)}\,d\tau,\end{aligned} \tag{6'}$$

where B, $C(z,\bar z)$, $F_1(z,\bar z,t)$, $F_2(z,\bar z,\tau)$ are analytic functions of their arguments and are expressed through the coefficient A_1; moreover, from (5) it follows that if $w\in C^{(1,\alpha)}(D+L)$, then $\varphi(z)\in C^{(2,\alpha)}(D+L)$, $\psi(z)\in C^{(0,\alpha)}(D+L)$. Using the integral representations

$$\varphi(z) = \frac{1}{\pi i}\int_L \frac{\mu_1(t)\,dt}{t-z} + iC_1, \qquad \psi(z) = \frac{1}{\pi i}\int_L \frac{\mu_2(t)\,dt}{t-z} + iC_2, \tag{7}$$

where μ_1, μ_2 are real densities $\mu_1(t) \in C^{(2,\alpha)}(L)$, $\mu_2 \in C^{(0,\alpha)}(L)$, and C_1, C_2 are real constants [(4)], the solution of equation (3) can be written in the form

$$\begin{aligned} w(z,\bar z) = {} & \frac{1}{\pi i}\int_L \frac{\mu_2(t) + B(t,\bar t)\mu_1'(t) + \bar t\mu_1''}{t-z}\,dt - \frac{1}{\pi i}\int_L \frac{\bar t - \bar z}{t-z}\mu_1'(t)\,dt \\ & - \frac{1}{\pi i}\int_L \frac{B(z,\bar t) - B(z,\bar z)}{t-z}\mu_1'(t)\,dt - \frac{A_1(z,\bar z)}{\pi i}\int_L \overline{\frac{\mu_1(t)\,d\bar t}{\bar t - \bar z}} + \frac{C}{\pi i}\int_L \frac{\mu_1\,dt}{t-z} \\ & + \frac{1}{\pi i}\int_L \mu_1(t)\int_0^z \frac{F_1(z,\bar z,\xi)\,d\xi}{t-\xi}\,dt + \frac{1}{\pi i}\int_L \mu_1(\bar t)\int_0^{\bar z} \frac{F_2(z,\bar z,\eta)\,d\eta}{\bar t - \eta}\,d\bar t + iC_2. \end{aligned} \tag{8}$$

The first integral in formula (8) is a holomorphic function of class $C^{(0,\alpha)}(D+L)$ and can be represented by a Cauchy-type integral with real density $\nu(t) \in C^{(0,\alpha)}(L)$ in the form

$$\frac{1}{\pi i}\int_L \frac{\mu_2(t) + B(z,\bar t)\mu_1' + \bar t\mu_1''}{t-z}\,dt + iC_2 = \frac{1}{\pi i}\int_L \frac{\nu(t)\,dt}{t-z} + iC_3, \tag{9}$$

where $\nu(t)$ and C_3 are uniquely determined through μ_1, μ_2, C_2. Conversely, knowing μ_1, ν, C_3, one can uniquely determine μ_2, C_2.

Substituting (9) into formula (8), we obtain a representation of an arbitrary solution w of problem (2)–(3) in terms of two arbitrary real functions $\mu_1(t)$ and $\nu(t)$:

$$\begin{aligned} w(z,\bar z) = {} & \frac{1}{\pi i}\int_L \frac{\nu(t)\,dt}{t-z} + iC_3 + \frac{C(z,\bar z)}{\pi i}\int_L \frac{\mu_1(t)\,dt}{t-z} - \frac{A_1(z,\bar z)}{\pi i}\int_L \overline{\frac{\mu_1(t)\,dt}{\bar t - \bar z}} \\ & - \frac{1}{\pi i}\int_L \frac{\bar t - \bar z}{t-z}\mu_1'(t)\,dt - \frac{1}{\pi i}\int_L \frac{B(z,\bar t) - B(z,\bar z)}{t-z}\mu_1'(t)\,dt \\ & + \frac{1}{\pi i}\int_L \mu_1(t)\int_0^z \frac{F_1(z,\bar z,\xi)}{t-\xi}\,dt + \frac{1}{\pi i}\int_L \mu_1(\bar t)\int_0^{\bar z} \frac{F_2(z,\bar z,\eta)\,d\eta}{\bar t - \eta}\,d\bar t. \end{aligned} \tag{10}$$

Passing to the limit as $z \to t_0 \in L$, we obtain a singular integral equation of the form

$$\nu(t_0) + \frac{1}{\pi i}\int_L \frac{\nu(t)\,dt}{t-t_0} + A_1(t_0,\bar t_0)\mu_1(t_0) - \frac{A_1(t_0,\bar t_0)}{\pi i}\int_L \overline{\frac{\mu_1\,d\bar t}{\bar t - \bar t_0}} +$$

$$+C(t_0,\bar{t}_0)\mu_1(t_0)+\frac{C(t_0,\bar{t}_0)}{\pi i}\int_L \frac{\mu_1\,dt}{t-t_0}+\int_L K_1(t_0,\bar{t}_0,t)\mu_1(t)\,dt+$$

$$+\int_L K_2(t_0,\bar{t}_0,\bar{t})\mu_1(\bar{t})\,d\bar{t}=f(t_0)-iC_3, \qquad (11)$$

where the kernels K_1 and K_2 have at $t=t_0$ a singularity of order less than 1. Equating the real and imaginary parts of expression (11), we obtain a system of two equations with real coefficients. It is easy to see that, for $A_1(t_0,\bar{t}_0)\neq 0$, this system is of normal type (4).

Equation (11) is equivalent to problem (3)–(2), i.e., if problem (3)–(2) has a solution, then equation (11) is solvable, and, conversely, if equation (11) is solvable, then problem (3)–(2) has a solution.

Since $\varphi(0)=\varphi'(0)=0$, we obtain the conditions on $\mu_1(t)$

$$\frac{1}{\pi i}\int_L \frac{\mu_1(t)\,dt}{t}+iC_1=0, \qquad \frac{1}{\pi i}\int_L \frac{\mu_1(t)}{t^2}\,dt=0. \qquad (12)$$

Problem (3)–(2) is equivalent to equation (11) and to condition (12), which is used only in computing the index. We note that for $f\in C^{(2,\alpha)}(L)$ all Hölder-continuous solutions (μ_1,ν) of equation (11), for $A_1\neq 0$ on L, belong to the class $C^{(2,\alpha)}(L)$, and, according to formula (10), the solution of problem (2)–(3) belongs to the class $C^{(1,\alpha)}(D+L)$, but it may fail to belong not only to $C^{(2,\alpha)}(D+L)$, but even to $C^{(1,\beta)}(D+L)$ for $\beta>\alpha$. Indeed, in formula (10) all integrals, except the second, belong to $C^{(2,\alpha)}(D+L)$, while the second integral

$$\eta(x,y)=-\frac{1}{\pi i}\int_L \frac{\bar{t}-\bar{z}}{t-z}\,\mu_1''(t)\,dt$$

is a function of class $C^{(1,\alpha)}(D+L)$, when $\mu_1(t)\in C^{(2,\alpha)}(L)$ (4).

For the disk $|z|\leq 1$, when $A_1=\mathrm{const}\neq 0$, a more concrete result is obtained. The general representation (6) in this case has the form:

$$w(z,\bar{z})=\psi(z)+\bar{z}\varphi(z)+A_1\int_0^{\bar{z}}\overline{(z-t)\varphi(t)}\,dt+$$

$$+|A_1|^2\int_0^z \varphi(t)\left\{\int_0^z d\tau_1\int_0^{\tau_1} I_0(2|A_1|(z-t)^{1/2}(\tau_1-\tau)^{1/2})\,d\tau\right\}dt+$$

$$+A_1|A_1|^2 \int_0^{\bar z} d\xi \int_0^{\xi} d\tau \left[\int_0^{z} I_0(2|A_1|(z-t)^{1/2}(\xi-\tau)^{1/2})\, dt \cdot \int_0^{\bar\tau} \overline{\varphi(\eta)}\, d\eta \right], \tag{6''}$$

where I_0 is the Bessel function of zero order of imaginary argument.

Expanding the functions $\varphi(z)$ and $\psi(z)$ in Taylor series,

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad \psi(z) = \sum_{n=0}^{\infty} b_n z^n,$$

and $f(\theta)$ in a Fourier series,

$$f(\theta) = \sum_{n=-\infty}^{+\infty} f_n e^{in\theta}, \qquad \text{where} \quad f_n = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{in\theta}\, d\theta,$$

and sub-

substituting into formula (6), for $|z| = 1$ we obtain a system of linear equations for determining the coefficients $a_n, b_n,\ n \geqslant 0$

$$I_1(2|A_1| - 1)a_0 = f_1, \qquad A_1^{-1} I_2(2|A_1|)a_0 = \overline{f}_{-2},$$

$$|A_1|^{-n-2}\overline{A}_1 n! I_{n+2}(2|A_1|)a_n = \overline{f}_{-(n+2)} \qquad (n \geqslant 1),$$

$$b_n + a_{n+1}[|A_1|^{-n-2}(n+1)! I_n(2|A_1|) - (n+1)!|A_1|^{-2} - 1] = f_n \quad (n \geqslant 0),$$

where

$$I_n(2|A_1|) = \sum_{k=0}^{\infty} \frac{|A_1|^{2k+n}}{k!(k+n)!},$$

whence we obtain:

Theorem 2. The homogeneous problem (3′)–(2) for $A_1 = \text{const} \neq 0$ in the disk $|z| \leqslant 1$ has only the trivial solution, and the nonhomogeneous problem is solvable under the condition

$$I_2(2|A_1|) \int_0^{2\pi} \overline{f}(\theta) e^{i\theta}\, d\theta = \overline{A}_1 (I_1(2|A_1|) - 1) \int_0^{2\pi} f(\theta) e^{-2i\theta}\, d\theta.$$

By analogous methods one obtains:

Theorem 3. The Dirichlet problem for the equation

$$w_{z\overline{z}} - (A_3\overline{w})_{\overline{z}} = 0 \tag{13}$$

is Noetherian if

$$A_3(z,\overline{z}) \neq 0 \quad \text{on } L,$$

and the index $\varkappa$ of the problem is equal to

$$\frac{1}{\pi}\left[\arg A_3(z,\overline{z})\right]_L + 2.$$

Theorem 4. The homogeneous problem (13)–(2) for $A_3 = \text{const} \neq 0$ in the disk $|z| \leqslant 1$ has two nontrivial linearly independent solutions over the field of real numbers, and the nonhomogeneous problem is unconditionally solvable for $f \in C^{(2,\alpha)}(L)$.

Theorem 5. The Dirichlet problem for the equation

$$w_{z\overline{z}} - A_5\overline{w} = 0 \tag{14}$$

is Noetherian if $A_5(z,\overline{z}) \neq 0$ on L and $f \in C^{(3,\alpha)}(L)$, and the index of the problem is equal to

$$\frac{1}{\pi}\left[A_5(z,\overline{z})\right]_L.$$

It is easy to see that the Dirichlet problem for the equation $w_{z\overline{z}} - A_4 w_z = 0$, $A_4 = \text{const} \neq 0$, in the disk $|z| \leqslant 1$, is not Noetherian, since the general representation in this case has the form $w = \psi(z) + \varphi(z)e^{A_4 z}$, where φ, ψ are arbitrary holomorphic functions in the disk.

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Note: Figure translations are in progress. See original paper for figures.