ON THE MOMENTUM AND ENERGY OF GRAVITATIONAL WAVES

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ON THE MOMENTUM AND ENERGY OF GRAVITATIONAL WAVES

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As is known, the integral conservation laws for momentum, energy, and mass in the general theory of relativity have meaning only for regions of space-time with Galilean conditions at infinity and can be formulated in the form

$$P^{\mu} = \int (g)(T^{\mu 4} + t^{\mu 4})\, dV, \tag{1}$$

where $T^{\mu 4}$ and $t^{\mu 4}$ are the tensor and pseudotensor of the energy-momentum of matter and of the gravitational field, P^{μ} is the four-dimensional momentum vector of this region, and dV is the element of spatial volume (^{1}).

We shall take $x^4 = ict$ and write Einstein' s law of gravitation in the form

$$R_{\mu\nu} - {}^1/{}_2 g_{\mu\nu} R = 8\pi T_{\mu\nu}. \tag{2}$$

Then (^{1}, ^{2})

$$(g)(T^{\mu\nu} + t^{\mu\nu}) = U^{\mu\nu}, \tag{3}$$

where

$$U^{\mu\nu} = \frac{1}{8\pi} \frac{\partial^2}{\partial x^{\alpha} \partial x^{\beta}} \left(\bar{g}^{\alpha\beta} \bar{g}^{\mu\nu} - \bar{g}^{\alpha\mu} \bar{g}^{\beta\nu} \right), \tag{4}$$

where $\bar{g}^{\alpha\beta} = \sqrt{g} g^{\alpha\beta}$, $g^{\alpha\beta}$ is the metric tensor.

The pseudotensor $t^{\mu\nu}$ at a given point of space-time can always be reduced to zero by a corresponding coordinate transformation, and, conversely, in Galilean space-time even purely spatial coordinate transformations can lead to values $t^{\mu\nu} \neq 0$.

Thus, localization of the momentum-energy of the gravitational field is devoid of physical meaning, at least in the sense in which we use it as applied to other fields and particles.

In contrast to this, the integral energy-momentum vector of a purely gravitational field, determined by solutions of the equations of gravitation

$$R_{\mu\nu} = 0 \tag{2a}$$

with

$$R_{\mu\nu\sigma} \neq 0 \tag{5}$$

and with Galilean conditions at infinity, has a completely definite meaning, and therefore its computation by formula (1) is of undoubted interest for some particular case. It is also essential that the quantity P^{μ} is covariant with respect to any coordinate transformations that become Galilean at infinitely distant points. If the quantity (1) for a purely gravitational field without matter turned out to be different from zero, then the gravitational field concentrated in some region of space-time would possess a certain material essence, similar to that possessed by other fields and particles occupying some region of space-time.

Here we shall confine ourselves to analyzing wave-type solutions for a weak field in the form

$$g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}, \tag{6}$$

where $|h_{\mu\nu}| \ll 1$ are small quantities, together with their derivatives. In this case, as is known ($^{1-3}$), putting

$$h_{\mu\nu} = \varphi_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\varphi_{\alpha\alpha} \tag{7}$$

and taking the coordinate conditions in the form

$$\partial\varphi_{\mu\sigma}/\partial x^{\sigma} = 0, \tag{8}$$

equations (2a) can be written as

$$\partial^2 h_{\mu\nu}/\partial x^{\sigma}\partial x^{\sigma} = 0. \tag{9}$$

One of the simplest solutions of (9) will be

$$h_{\mu\nu} = f_{\mu\nu}(\xi)/r \tag{10}$$

where

$$\xi = r + ix_4, \qquad r = \sqrt{x_1^2 + x_2^2 + x_3^2}; \tag{11}$$

$f_{\mu\nu}$ are arbitrary functions of the parameter ξ. In order that the $h_{\mu\nu}$ be solutions of (9), they must also satisfy the coordinate conditions (8). Therefore, of the 10 quantities $h_{\mu\nu}$, only 6 can contain arbitrary functions of ξ; the remaining 4 must be determined by the coordinate conditions (8). As these latter we choose $h_{11}, h_{22}, h_{33}, h_{44}$.

These quantities will satisfy equations (8) in the form

$$h_{(\alpha\alpha)} = \int_0^{x_{(\alpha)}} \frac{\partial h_{(\alpha)\sigma}}{\partial x^\sigma}\, dx_{(\alpha)} - \frac{1}{2}\delta_{(\alpha\alpha)} \int_0^{x_\beta} \frac{\partial h_{\beta\sigma}}{\partial x^\sigma}\, dx_\beta, \tag{12}$$

$$(\sigma \neq \alpha) \qquad\qquad\qquad\qquad (\sigma \neq \beta)$$

where there is no summation over the indices α in parentheses.

Let us note that $h_{(\alpha\alpha)}$ are quantities of the same order as $h_{\mu\nu}$ for $\mu \neq \nu$, as is seen from equations (8) and (12).

Thus, solutions have been found for all $h_{\mu\nu}$ in the form (10) for $\mu \neq \nu$ and (12) for $\mu = \nu$, satisfying the wave equations (9) and the coordinate conditions (8). These $h_{\mu\nu}$, therefore, will be solutions of the gravitational equations (2a).

In accordance with (4)

$$P^\mu = \frac{1}{8\pi} \int \frac{\partial}{\partial x^\beta} \left(\overline{g}^{\alpha 3} \overline{g}^{\mu 4} - \overline{g}^{\alpha\mu} \overline{g}^{34} \right) dS_\alpha. \tag{13}$$

The integration here is carried out over a purely spatial surface S, located at points infinitely remote from the origin of coordinates, on which the Galilean condition of space-time is fulfilled.

From relation (13) it is seen that each of the functions under the integral contains, as factors, derivatives of the form $\partial h_{\mu\nu}/\partial x^\sigma$ and $\partial^2 h_{\mu\nu}/\partial x^\sigma \partial x^\gamma$, and moreover the terms containing second derivatives, according to (12), will also contain an additional integration with respect to the coordinate x_α, not changing the order of magnitude. This is because the quantities $\partial h_{\mu\nu}/\partial x^\sigma$ for $\mu \neq \nu$ and integrals of the form

$$\int \frac{\partial^2 h_{\mu\nu}}{\partial x^\sigma \partial x^\gamma}$$

may be regarded as having the same order of magnitude as the derivatives $\partial h_{\mu\nu}/\partial x^\gamma$; this is sufficiently obvious in the case $\sigma = \gamma$.

Fig. 1

Figure 1: Fig. 1

For the moment $x_4 = 0$, the functions $f_{\mu\nu}(\xi) = f_{\mu\nu}(r)$, owing to their arbitrariness, can always be chosen so that they vanish, together with all their derivatives, not only on the boundaries of infinitely remote-

closed surface S, but even at distances r from the origin of coordinates greater than some finite value r_0. What has been said can also be applied to the quantities $h_{\mu\nu}$ and their derivatives. The properties of solutions of this kind can be illustrated by the graphs (Fig. 1).

In this case the wave formations described by the solutions (10), which we shall call wave packets, will have a four-dimensional energy-momentum vector (13) equal to zero. It can be shown that for solutions of this kind the functions $f_{\mu\nu}$ can always be chosen so that the Riemann-Christoffel curvature tensor is nonzero. For example, for $x_2 = x_3 = 0$,

Fig. 1

$$R_{1234} = \frac{1}{2}\left(\frac{\partial^2 h_{14}}{\partial x^2 \partial x^3} + \frac{\partial^2 h_{23}}{\partial x^1 \partial x^4} - \frac{\partial^2 h_{13}}{\partial x^2 \partial x^4} - \frac{\partial^2 h_{24}}{\partial x^1 \partial x^3}\right) =$$

$$= \frac{1}{2} i x_1 \left(\frac{f''_{23}}{r^2} - \frac{f'_{23}}{r^3}\right) \neq 0 \tag{14}$$

with a suitable choice of $f_{23}(\xi)$. The prime denotes differentiation with respect to ξ.

It may seem that the vanishing of P^μ occurs because of the choice of a gravitational wave packet in the form (10), under the condition that $f_{\mu\nu}$ and its derivatives vanish at distances $r > r_0$.

In fact this is not so, as can be verified by calculations for an electromagnetic wave packet of the same type within the framework of the special theory of relativity.

Suppose that the 4-vector potentials are given in the form

$$A_1 = \frac{f(r + ix_4)}{r}, \qquad A_2 = A_3 = 0, \qquad A_4 = \frac{if(r + ix_4)}{r}, \tag{15}$$

and, obviously, satisfy an equation of type (9).

Calculating, with the help of this solution, for example, P^4, we find

$$P^4 = \int_V T^{44}\, dV = \int_V -\frac{1}{4\pi}\left\{\left(\frac{x_2^2}{r^4} + \frac{x_3^2}{r^4} + \frac{4x_1^2}{3r^4}\right)\left(\frac{f}{r} - f'\right)^2 + \frac{4}{3}\frac{{f'}^2}{r^2}\right\} dV \neq 0. \tag{16}$$

Hence it is clear that, by setting f and f' equal to zero beginning at some $r > r_0$, it is impossible to make the quantity P^4 vanish; i.e., the vanishing of P^4 for gravitational waves is by no means connected with such a choice of solution.

These solutions describe wave packets of gravitational waves in the absence of sources of the field and are analogous to wave packets of electromagnetic waves occupying, at the given moment of time, a finite region of space containing no charged or neutral particles of matter. In electrodynamics such packets may be called photons, and in the theory of gravitation—gravitons.

The vanishing of the vector P^4 for a wave packet of this kind makes it possible to conclude that gravitons defined in this way cannot contribute to the energy-momentum balance in the transmutation of particles, for example in the transmutation of an electron-positron pair into gravitons. This result remains unchanged if the calculations are carried out up to terms of second order. By the method of successive approximations one can find an approximation of higher order of smallness as well. To do this one must substitute $h_{\mu\nu}$ in the first approximation into the equation of gravitation (2) and regard the terms of second order of smallness as the source of gravitational waves in the second approximation. This source, accurate to terms of second order, will contain a sum of expressions of the following form with numerical multipliers:

$$\frac{\partial h_\alpha}{\partial x^\gamma}\frac{\partial h_{\mu\nu}}{\partial x^\sigma}, \tag{17}$$

which, for the instant $x_4 = 0$ and, consequently, for all $x_4 > 0$, by virtue of the properties of the solutions in the form (10) of the first approximation, will not contribute, according to (13), to the energy-momentum of the gravitational wave packet.

It is not difficult to see that what has been said also applies to all higher-order approximations: the third, fourth, etc.

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Note: Figure translations are in progress. See original paper for figures.

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