ON THE SOLVABILITY OF NONLINEAR OPERATOR EQUATIONS IN BANACH SPACE

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 517.945.9

MATHEMATICS

G. N. AGAEV

ON THE SOLVABILITY OF NONLINEAR OPERATOR EQUATIONS IN BANACH SPACE

(Presented by Academician S. L. Sobolev on 27 VIII 1966)

The theory of the first boundary-value problem for quasilinear strongly elliptic and parabolic differential equations of higher order, having divergence form, was developed by M. I. Vishik ([4], [5]). For monotone elliptic equations this theory was developed in the works of F. Browder ([2], [3]). Yu. A. Dubinskii ([6], [7]) gave a simple proof of the theorem of M. I. Vishik and F. Browder, and also obtained a number of new results concerning quasilinear parabolic equations having nondivergence form, and degenerate quasilinear elliptic and parabolic equations.

The aim of the present note is to establish the solvability of nonlinear operator equations in reflexive separable Banach spaces. Such an approach makes it possible, in a unified way, to cover a broad class of boundary-value problems for quasilinear elliptic and parabolic equations, as well as equations that are neither strongly elliptic nor monotone, nonlinear integro-differential equations, and other types of equations.

1. Let X and Y be reflexive Banach spaces, with X separable. Denote the corresponding dual spaces by X^* and Y^*. Consider operators K_1, K_2, K_3, defined as follows:

1°. K_1 is linear and maps Y into X^*. Then its adjoint K_1^* will map X into Y^*. Suppose that K_1^* is demicontinuous in the sense of F. Browder ([2]), i.e., it carries every strongly convergent sequence to $u \in X$ into a sequence weakly convergent to $K_1^* u \in Y^*$.

2°. $K_2(u)$, generally speaking nonlinear, maps X into Y and is weakly continuous, i.e., it carries every sequence weakly convergent to $u \in X$ into a sequence weakly convergent to $K_2(u) \in Y$.

3°. $K_3(u)$, generally speaking nonlinear, is weakly continuous and maps X into X^*.

4°. There exist γ, χ, independent of $u \in X$, such that the "ellipticity" inequality is satisfied

$$(K_2(u), K_1^* u) + (K_3(u), u) \geq \gamma \|u\|_X^p - \chi, \tag{1}$$

for arbitrary $u \in X$. Here (y, x) denotes the value of the functional y at the point x, $p > 1$.

One of the sufficient conditions under which inequality (1) holds is given by the inequalities

$$(K_2(u), K_1^* u) \geq \gamma \|u\|_X^p - \chi; \tag{2}$$

$$\|K_3(u)\|_{X^*} \leq \eta \|u\|_X^{p-1}, \tag{3}$$

where η is sufficiently small. Conditions (2), (3) are satisfied, for example, for elliptic differential operators in domains of sufficiently small diameter.

Consider an operator equation of the form

$$K_1 K_2(u) + K_3(u) = h, \tag{4}$$

where $h \in X^*$ is given arbitrarily. We shall investigate the solvability of equation (4) in the class of weak solutions by the Galerkin method.

By a weak solution of equation (4) we mean an element $u \in X$ for which the relation

$$(K_2(u), K_1^* v) + (K_3(u), v) = (h, v) \tag{5}$$

holds for every $v \in X$.

Let the system of elements $\{v\}$ be complete in X; let V^k be the subspace spanned by the first k elements $\{v\}$. We seek an approximate solution u_k of equation (4) in the space V^k from the relations

$$(K_1 K_2(u_k), v) + (K_3(u_k), v) = (h, v), \qquad v \in V^k. \tag{6}$$

The existence of a solution of equation (6) is established on the basis of the following lemma, which is an abstract analogue of the lemma of M. I. Vishik ([5]).

Lemma (see ([8])). Let V be a finite-dimensional space, V^* its conjugate. Let A be a finite-dimensional, generally speaking nonlinear, operator acting from V into V^*, for which the inequality

$$(A(v), v) \geqslant \gamma \|v\|^p - \chi, \qquad p > 1,$$

holds for every $v \in V$, where γ and χ are some positive constants. Then for every $h \in X^*$ there exists at least one element $v \in V$ such that $A(v) = h$.

In our case $A = K_1 K_2 + K_3$.

We now establish a priori estimates.

Relations (6), for the found $u_k \in V^k$ and $v = u_k$, give

$$(K_2(u_k), K_1^* u_k) + (K_3(u_k), u_k) = (h, u_k).$$

Applying Young' s inequality, we obtain

$$\gamma \|u_k\|_X^p - \chi \leqslant \frac{\|h\|_{X^*}^q}{q\varepsilon^q} + \frac{\varepsilon^p}{p}\|u_k\|_X^p, \qquad q = \frac{p}{p-1},$$

where ε is an arbitrary positive number. Choosing it sufficiently small, we obtain the uniform boundedness of all Galerkin approximations:

$$\|u_k\|_X \leqslant M.$$

Since every ball in a reflexive Banach space is weakly compact, from the sequence $\{u_k\}$ one can extract a subsequence $\{u_m\}$ weakly convergent in X. Denote its weak limit by u. We shall prove that u is the desired solution.

We rewrite system (6) in the form

$$(K_2(u_m), K_1^* v) + (K_3(u_m), v) = (h, v) \tag{7}$$

for $v \in V^l$, where $l > 0$ is an arbitrary fixed number, and $m \geqslant l$. Fix $v \in V^l$ in (7) and pass to the limit as $m \to \infty$. In view of the weak continuity of the operators K_2 and K_3, we obtain

$$(K_2(u), K_1^* v) + (K_3(u), v) = (h, v), \qquad v \in V^l. \tag{8}$$

Since the union $\bigcup_{l=1}^{\infty} V^l$ is dense in X, it follows, by the hemicontinuity of K_1^*, that relation (8) remains valid also for every $v \in X$. Consequently, u is the desired solution. Thus, the following holds.

Theorem 1. Let the operators K_1, K_2, K_3 satisfy conditions 1°–4°. Then the operator equation (4), for every right-hand side $h \in X^*$, has at least one weak solution in the reflexive separable Banach space X, and this solution can be obtained as the weak limit of certain Galerkin approximations.

2. In Theorem 1 we subjected the operators to conditions 1^0–4^0. In various applications the listed conditions need to be weakened. For this purpose, for example, we shall replace the weak continuity of the operators K_2, K_3 by a more general condition, namely, their semicontinuity. At the same time, in order to establish the solvability of equation (4), the boundedness of K_2, K_3 will also be required (bounded sets are mapped into bounded sets). In addition to inequality (1), it is required that additional inequalities connecting $K_i,\ i = 1, 2, 3$, be satisfied. For example, for all $u, v \in X$ the monotonicity condition is fulfilled

$$(K_2(u) - K_2(v),\, K_1^*(u - v)) + (K_3(u) - K_3(v),\, u - v) \geq 0. \tag{9}$$

We shall still define a weak solution of equation (4) by relation (5), and approximate solutions by (6).

Repeating the indicated scheme, we obtain that all Galerkin approximations are uniformly bounded, $\|u_k\| \leq M$.

Hence, by virtue of the compactness of the sphere in X, we can select a subsequence $\{u_r\}$ converging to some $u \in X$. Since the operators K_2, K_3 are bounded, $\|K_2(u_r)\|_Y,\ \|K_3(u_r)\|_{X^*} \leq M_1$, where M_1 is some positive number. Hence one can select a subsequence $\{u_m\}$ for which $K_2(u_m)$ will converge weakly to some element w in Y, and $K_3(u_m)$ to ω weakly in X^*.

Writing relations (7) for u_m, fixing $v \in V^l,\ m > l$, and passing to the limit as $m \to \infty$, we obtain

$$(w, K^*v) + (\omega, v) = (h, v). \tag{10}$$

It is clear that this relation also remains valid for any $v \in X$. Further, using the monotonicity (9) of the operator $K_1K_2 + K_3$ and the semicontinuity of K_2, K_3, it is easy to prove the validity of the identity

$$(w, K_1^*z) + (\omega, z) \equiv (K_2(u), K_1^*z)$$

for any $z \in X$. Hence, by virtue of (10), we obtain that u is the desired solution. Thus, we have

Theorem 2. *Let X, Y be two reflexive Banach spaces, with X separable. Let K_1^* be linear, semicontinuous, and act from X into Y^*; let the operators K_2, K_3 be bounded and semicontinuous, with K_2 acting from X into Y, and K_3 from X into X^*, and suppose that inequalities (1) and (9) are satisfied for them. Then equation (4) has at least one weak solution. This solution can be obtained as the weak limit of a certain sequence of Galerkin approximations.*

Next we note that the weak solution of the equation will be unique if the "strong ellipticity" condition is satisfied:

$$(K_2(u) - K_2(v),\, K_1^*(u-v)) + (K_3(u) - K_3(v),\, u-v) \geq \gamma \|u-v\|^p, \qquad (11)$$

where $\gamma > 0$ and does not depend on $u, v \in X$.

In this case the operator $K = K_1 K_2 + K_3$ has a unique inverse K^{-1}. It is also easy to prove its continuity. Let $h \in X^*$, $h_\nu \in X^*$, and suppose that $\|h - h_\nu\|_{X^*} \to 0$ as $\nu \to \infty$. The weak solutions of equation (4) corresponding to its right-hand sides h and h_ν will be denoted by u and u_ν, respectively. Then we have the identity

$$(K_2(u) - K_2(u_\nu),\, K_1^*(u-u_\nu)) + (K_3(u) - K_3(u_\nu),\, u-u_\nu) = (h - h_\nu,\, u - u_\nu).$$

Apply inequality (11) to its left-hand side, and the generalized Schwarz inequality to the right-hand side. Then we obtain

$$\|u - u_\nu\|_X^{p-1} \leq C \|h - h_\nu\|_{X^*}.$$

Hence we obtain the required result, and consequently the following is proved.

Theorem 3. *Under the conditions of Theorem 2 and inequality (11), the mapping $K : X \to X^*$ is a homeomorphism.*

To illustrate the general results obtained, let us consider the first boundary-value problem for the equation

$$\sum_{i=1}^{n} (-1)^{m_i} \frac{\partial^{m_i}}{\partial x_i^{m_i}} \left(a_i(x) \left| \frac{\partial^{m_i} u}{\partial x_i^{m_i}} \right|^{p_i - 1} \operatorname{sign} \frac{\partial^{m_i} u}{\partial x_i^{m_i}} \right) + A(u) = h(x) \qquad (12)$$

in the space $\dot{W}_p^{(\mathbf{m})}(G)$, where $\mathbf{m} = (m_1, m_2, \dots, m_n)$; $\mathbf{p} = (p_1, p_2, \dots, p_n)$; G is a certain bounded domain in the n-dimensional Euclidean space R^n (see (9)); $h(x) \in W_q^{(-\mathbf{m})}(G)$; $a_i(x)$ are continuous in G; $|a_i(x)| \geq a_0 > 0$; $q_i = p_i/(p_i - 1)$; $A(u)$ is a certain weakly continuous operator; $\|A(u)\|_{W_q^{(-\mathbf{m})}} \leq \eta \|u\|_X^{p_0 - 1}$; $\eta > 0$ is sufficiently small; $p_0 = \min(p_1, p_2, \dots, p_n)$.

The solution of problem (12) is easily reduced to the solution of the operator equation (4), if one introduces the vector-operators

$$K_1 = (K_1^1, K_2^1, \dots, K_n^1), \qquad K_2 = (K_1^2, K_2^2, \dots, K_n^2),$$

where

$$K_i^1 = (-1)^{m_i} \frac{\partial^{m_i}}{\partial x_i^{m_i}}, \qquad K_i^2 u = a_i(x) \left| \frac{\partial^{m_i} u}{\partial x_i^{m_i}} \right|^{p_i - 1} \operatorname{sign} \frac{\partial^{m_i} u}{\partial x_i^{m_i}},$$

$i = 1, 2, \dots, n$. We define the action of these vector-operators by the formula

$$(K_2(u), K_1^* v) = \sum_{i=1}^{n} (K_i^2(u), K_i^{1*} v),$$

and define a weak solution of problem (12) from relation (5).

It is easy to verify that the conditions of Theorem 1 are satisfied. Let us verify, for example, the validity of inequality (2):

$$(K_2(u), K_1^* u) = \sum_{i=1}^{n} \left(a_i(x) \left| \frac{\partial^{m_i} u}{\partial x_i^{m_i}} \right|^{p_i - 1}, \left| \frac{\partial^{m_i} u}{\partial x_i^{m_i}} \right| \right) \geq$$

$$\geq a_0 \sum_{i=1}^{n} \int_G \left| \frac{\partial^{m_i} u}{\partial x_i^{m_i}} \right|^{p_i} dx \geq C_0 \|u\|^{p_0}_{\dot{W}_p^{(\mathbf{m})}(G)} - \varkappa_0,$$

where $C_0, \varkappa_0$ are certain constants. The conditions of weak continuity of the operators K_i^{1*} (and therefore also of K_1^*), and of semicontinuity of K_i^2 (and therefore also of K_2) are likewise easily verified. Examples can be given of differential operators that are not "strongly elliptic," are not monotone, but satisfy the conditions of Theorem 1 (see (1)).

Institute of Mathematics and Mechanics
Academy of Sciences of the Azerbaijan SSR

Received
26 VIII 1966

REFERENCES

1. G. N. Agaev, *Izv. AN AzerbSSR, Ser. Phys.-Techn. and Math. Sciences*, No. 5 (1966).

2. F. E. Browder, *Nonlinear elliptic boundary-value problems*, Materials for the Soviet-American Symposium on Partial Differential Equations, August 1963, Novosibirsk.

3. F. E. Browder, *Bull. Am. Math. Soc.*, **69**, 862 (1963).

4. M. I. Vishik, *Mat. sbornik*, **19** (101), 289 (1962).

5. M. I. Vishik, *Tr. Moscow Math. Soc.*, **12**, 128 (1963).

6. Yu. A. Dubinskii, *Mat. sbornik*, **67** (109), no. 4 (1965).

7. Yu. A. Dubinskii, *Dokl. Scientific-Technical Conf. on the Results of Scientific Research for 1964–1965, Mathematics Section*, Moscow Power Engineering Institute, 1965.

8. J. Leray, L. Lions, *Bull. Soc. Math. France*, **93**, 97 (1965).

9. S. M. Nikol' skii, *UMN*, **16**, No. 6, 63 (1961).

Note: Figure translations are in progress. See original paper for figures.