ON APPROXIMATIONS OF ANALYTIC FUNCTIONS BY RATIONAL FUNCTIONS

A. L. LEVIN, V. M. TIKHOMIROV

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Abstract

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A. L. LEVIN, V. M. TIKHOMIROV

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Let B be a Banach space of functions $f(z)$ of a complex variable z; $\|f\|$ the norm of f in B; $\mathcal{P}_n$ the linear space of polynomials $P_n(z) = a_0 + a_1 z + \cdots + a_{nz}^n$ of degree $\leq n$; $\mathcal{R}_n$ the set of all rational functions $R_n(z) = P_{1k}(z)/P_{2s}(z)$, $\max(k, s) \leq n$.

Put

$$e_n(f) = \inf_{P_n \in \mathcal{P}_n} \|f - P_n\|, \qquad r_n(f) = \inf_{R_n \in \mathcal{R}_n} \|f - R_n\|.$$

In this paper we mainly take as B the space A^1_∞ of functions $f(z)$, analytic for $|z| < 1$ and continuous in the closed disk $|z| \leq 1$, with norm $\|f\| = \max_{|z|=1} |f(z)|$. Our concrete results follow from the following elementary lemma of functional analysis.

Lemma. Let B be any Banach space and L its subspace. Let $\varphi_0 \in L$ be an element of best approximation to a given $f_0 \in L$. Then we have:

$$\|f_0 - \varphi_0\| = \max_{\substack{F \in B^* \\ F \perp L}} |F(f_0)|/\|F\|,$$

and the maximum is attained.

Indeed, since

$$|F(f_0)| = |F(f_0 - \varphi)| \leq \|F\| \cdot \|f_0 - \varphi\|,$$

we have

$$\|f_0 - \varphi\| \geq |F(f_0)|/\|F\| \quad \forall\, \varphi \in L.$$

At the same time there exists a functional F such that

$$\|f_0 - \varphi_0\| \cdot \|F\| = |F(f_0)|$$

(see ([3])). This is an obvious consequence of the Hahn-Banach theorem. Conversely, if $F \perp L$ and

$$|F(f_0)| = \|F\| \cdot \|f_0 - \varphi_0\|,$$

then we have

$$\|f_0-\varphi_0\| = |F(f_0)|/\|F\| = |F(f_0-\varphi)|/\|F\| \le \|f_0-\varphi\|, \qquad \forall\varphi\in L,$$

and φ_0 is an element of best approximation.

We pass to approximations by rational functions. In this paper, continuing ([1]), we carry out a comparison of the apparatus of approximation by rational functions and by polynomials.

Theorem 1. Put

$$f(z)=\sum_{m=0}^{\infty} c_m z^{\lambda_m},$$

where $c_m>0$,

$$\sum_{m=0}^{\infty} c_m<\infty$$

and λ_{m+1}/λ_m is nonintegral. Let n be a natural number and let m_n be the first index such that

$$\lambda_{m_n-1}-\lambda_{m_n-2}\le n<\lambda_{m_n}-\lambda_{m_n-1}.$$

Then

$$r_n(f)\ge \sum_{m>m_n} c_m.$$

We note that in ([1]) the same function is considered, and for approximation in the norm L_2 the estimate

$$r_n\ge c_{m_n}/2n$$

is obtained. From the theorem just stated it follows at once that if

$$\lambda_{m_n-1}\le n<\lambda_{m_n}-\lambda_{m_n-1},$$

then

$$r_n(f)=e_n(f),$$

since, obviously,

$$e_n(f)\le \sum_{m\ge m_n} c_m.$$

Proof of Theorem 1. Fix the poles $a_1,\dots,a_n$ of the rational function $R_n(z)$. Put

$$\mu_k=\exp(\pi ik)\prod_{l=1}^{n}\left(\exp\left(\frac{\pi ik}{\lambda_{m_n}}\right)-\frac{1}{\overline{a_l}}\right)\left(\exp\left(-\frac{\pi ik}{\lambda_{m_n}}\right)-\frac{1}{a_l}\right).$$

For any $f\in A^1_\infty$ put

$$F(f) = \sum_{k=0}^{2\lambda_{m_n}-1} f(z_k)\overline{\mu_k}, \quad \text{where} \quad z_k = \exp\left(\frac{\pi i k}{\lambda_{m_n}}\right).$$

F is a linear continuous functional on A^1_∞. Let us clarify its properties.

$$F(1) = \sum \overline{\mu_k} = \overline{\sum \mu_k} = 0,$$

since the sum $\sum \mu_k$ contains terms of the form

$$\text{const}(a_1, \dots, a_n) \sum_{k=0}^{2\lambda_{m_n}-1} \exp\left(\frac{\pi i k}{\lambda_{m_n}} s\right) \exp(\pi i k), \qquad (*)$$

which is equal to zero if

$$\exp(\pi i) \exp\left(\frac{\pi i}{\lambda_{m_n}} s\right) \neq 1.$$

But this is so, because

$$-n \leq s \leq n, \quad \text{and} \quad n < \lambda_{m_n}.$$

Further,

$$F\left(\frac{1}{z - a_l}\right) = \sum_k \frac{\overline{\mu_k}}{z_k - a_l} = -\frac{1}{a_l^2} \sum \frac{\overline{\mu_k}}{z_k - 1/\overline{a_l}} = 0 \quad (l = 1, \dots, n),$$

because μ_k contains the factor $z_k - 1/\overline{a_l}$, and after canceling it we again obtain terms of the form $(*)$.

Finally,

$$F(z^p) = \sum_{k=0}^{2\lambda_{m_n}-1} \overline{\mu_k} z_k^p.$$

This sum contains terms of the form

$$\sum_{k=0}^{2\lambda_{m_n}-1} \exp(-\pi i k) \exp\left(\frac{\pi i k}{\lambda_{m_n}} s\right) \exp\left(\frac{\pi i k}{\lambda_{m_n}} p\right).$$

We want $F(z^p) = 0$ for $0 \leq p \leq \lambda_{m_n - 1}$. For this, as above, we must have

$$s + p < \lambda_{m_n}$$

or

$$s < \lambda_{m_n} - p \leq \lambda_{m_n} - \lambda_{m_n - 1}.$$

But this is satisfied by assumption. It remains to note that

$$F\left(z^{\lambda_{m_n}}\right) = \sum_{k=0}^{2\lambda_{m_n}-1} \prod_{l=1}^{n} \left| \exp\left(\frac{\pi i k}{\lambda_{m_n}}\right) - \frac{1}{a_l} \right|^2 = \|F\|$$

and even

$$F\left(z^{\lambda_m}\right) = \|F\|$$

for $m \geq m_n$ (because of the evenness of λ_{m+1}/λ_m). By the lemma,

$$r_n(f) \geq |F(f)|/\|F\| = \sum_{m \geq m_n} c_m,$$

as was required.

Corollary (see (1)). If the function considered $f(z) \in A_R^1$, i.e.

$$c_m \sim 1/R^m, \quad R > 1,$$

and if

$$\lambda_{m+1}/\lambda_m \to \infty,$$

then we have:

$$\overline{\lim_{n\to\infty}}\, r_n(f)^{1/n} = \overline{\lim_{n\to\infty}}\, e_n(f)^{1/n} = 1/R.$$

Theorem 1, in a somewhat different formulation and by other methods, was obtained by E. P. Dolzhenko in (2). The question arises whether Theorem 1 can be strengthened, i.e. whether there exists a function $f(z) \in A_\infty^1$ such that

$$e_m(f) = r_m(f), \quad m = 0, 1, \dots, n, \dots$$

The answer to this question is given by

Theorem 2. The functions $f(z) \equiv a_k z^k + b_k$, and only they, have the property that

$$e_n(f) = r_n(f), \quad \forall n = 0, 1, \dots$$

Theorem 2 follows from the following two theorems, which are of independent interest.

Theorem 3. Suppose it is known that the polynomial

$$P_k^0(z) = a_0 + a_1 z + \dots + a_{kz}^k, \quad a_k \neq 0,$$

is a polynomial of degree $k > 0$ of best approximation for $f(z)$, and $n+1$ is the first number exceeding k such that

$$e_k(f) = e_n(f) > e_{n+1}(f).$$

Then

$$e_n(f) > r_n(f).$$

Indeed, by the condition, the space $\mathcal{P}_n$ is supporting to the sphere of the space A^1_∞ of radius $\|f - P^0_k\|$ with center at $f - P^0_k$, but the space $\mathcal{P}_{n+1}$ is not supporting to it. This means that there exists a variation

$$t \cdot Q_{n+1}(z) \equiv t \cdot (\varepsilon'_0 + \varepsilon'_1 z + \dots + \varepsilon'_{n+1} z^{n+1}), \quad t > 0,$$

such that

$$\|f - P^0_k + t \cdot Q_{n+1}(z)\| = \|f - P^0_k\| - ta + o(t), \quad a > 0.$$

Consider the numbers $\varepsilon, \varepsilon_0, \varepsilon_1, \dots, \varepsilon_n$, defined by the equalities

$$\varepsilon a_k = \varepsilon'_{n+1}, \quad (\varepsilon a_{k-1} + \varepsilon_n) = \varepsilon'_n, \dots, \quad (\varepsilon a_0) + \varepsilon_{n-k+1} = \varepsilon'_{n-k+1}, \quad \varepsilon_{n-k} = \varepsilon'_{n-k}, \dots, \quad \varepsilon_0 = \varepsilon'_0.$$

Then we have

$$R_n(z;t) \equiv \frac{P^0_k(z) - t \cdot (\varepsilon_0 + \varepsilon_1 z + \dots + \varepsilon^n_{nz})}{1 - \varepsilon t z^{n-k+1}} = P^0_k(z) - t \cdot Q_{n+1}(z) + o(t).$$

Hence, as is easy to see, there is a $t_0 > 0$ such that

$$\|f(z) - R_n(z;t_0)\| = \|f(z) - P^0_k(z) + t_0 Q_{n+1}(z) - o(t_0)\| < \|f(z) - P^0_k(z)\|.$$

The theorem is proved.

Corollary. Let $e_0(f) > e_k(f) = \dots = e_n(f) > e_{n+1}(f)$. By Theorem 3 we have $r_n(f) < e_n(f)$. Hence, if $r_n(f) = e_n(f)\ \forall n$, then one must have

$$e_0(f) = \dots = e_{k-1} > e_k(f) = \dots = e_n(f) = \cdots.$$

But since $e_n(f) \to 0$ as $n \to \infty$, we obtain $e_k(f) = 0$, and hence $f(z)$ is a polynomial of degree k, and moreover such that $e_{k-1}(f) = e_0(f)$.

Theorem 4. Among all polynomials $P_k(z)$ of degree k, only for polynomials $P_k(z) \equiv a^k_{kz}$ do we have

$$e_{k-1}(P_k(z)) = \|P_k(z)\| = |a_k|. \qquad (**)$$

Indeed, for these polynomials condition $(**)$ is satisfied. Let us prove this. Fix the poles $\alpha_1, \dots, \alpha_{k-1}$. Put, for any $f(z) \in A^1_\infty$,

$$F(f) = \int_{|z|=1} f(z)\overline{\mu}(z)|dz|, \quad \text{where} \quad \mu(z) = z^k \prod_{i=1}^{k-1} (z - \alpha_i) \left(\frac{1 - \overline{\alpha}_i z}{z} \right).$$

F is obviously a linear continuous functional on A^1_∞. We have:

a)

$$\overline{z-\alpha_i} = \frac{1}{z} - \overline{\alpha_i}$$

on the circle $|z| = 1$, i.e.

$$\prod_{i=1}^{k-1}(z-\alpha_i)\left(\frac{1}{z} - \overline{\alpha_i}\right)$$

is real and positive.

b) $F \perp 1,\ 1/(z-\alpha_i),\ i = 1, \dots, k-1$, by Cauchy' s theorem, since $\mu(z)/z$ and $\mu(z)/(\overline{z} - \overline{\alpha_i}) = z\mu(z)/(1 - \alpha_i z)$ are analytic. (We note that $|dz| = dz/z$ if $|z| = 1$.)

c)

$$|F(a_{kz}^k)| = |a_k| \int_{|z|=1} \prod_{i=1}^{k-1}(z-\alpha_i)\left(\frac{1-\overline{\alpha_i}z}{z}\right)|dz| = \|F\|\,\|a_{kz}^k\|;$$

F satisfies the conditions of the lemma with $\varphi_0 = 0$, whence

$$\|a_{kz}^k\| \leq \left\| a_{kz}^k - c_0 - \sum_{i=1}^{k-1} \frac{c_i}{z-\alpha_i} \right\|.$$

In view of the arbitrariness of α_i and c_i, we obtain $r_n(a_{kz}^k) \geq |a_k|$, as was required.

This result was obtained by another method by V. M. Tikhomirov in 1962. We shall now prove that if the polynomial $P_k(z) \not\equiv a_{kz}^k$, then condition $(**)$ is not satisfied for it.

If for the polynomial $P_k(z) \not\equiv a_{kz}^k$ condition $(**)$ is satisfied, then, by the lemma, there exists a functional

$$F(f) = \int_{|z|=1} f(z)\,d\mu(z)$$

such that

$$\big|F(P_k(z))\big| = \left|\int_{|z|=1} P_k(z)\,d\mu(z)\right| = \max|P_k(z)| \int_{|z|=1} |d\mu(z)|, \tag{1}$$

$$F \perp z^s, \quad s = 0, 1, \dots, k-1. \tag{2}$$

In view of condition (1), the measure μ is concentrated on the set of points at which $|P_k(z)|$ attains its maximum. But this set consists of m points ($m \leq k$). Hence

$$F(f) = \sum_{i=1}^{m} f(\xi_i)\mu(\xi_i).$$

Condition (2) gives:

$$\sum_{i=1}^{m} \mu_i = 0, \qquad \sum_{i=1}^{m} \mu_i \xi_i^s = 0, \qquad s = 1, 2, \dots, k-1.$$

This system of k equations with m ($m \le k$) unknowns has only the trivial solution. The contradiction obtained proves Theorem 4. Theorem 4 can be strengthened in the following direction.

Theorem 4′. *If $f(z) \in A^1_\infty$, $|f(e^{i\theta})| = \|f\|$, $0 \le \theta \le 2\pi$, and in the disk $|z| < 1$ $f(z)$ has more than $k-1$ zeros, then $r_{k-1}(f) = e_{k-1}(f) = \|f\|$.*

We note that, by the maximum principle, the indicated class of functions is exhausted by Blaschke products.

Let K be a subset of A^1_∞. Denote

$$\varepsilon_n(K) = \sup_{f \in K} e_n(f), \quad \rho_n(K) = \sup_{f \in K} r_n(f).$$

Let B_r be the class considered in work ([4]).

$$B_r = \{f(z) : f(z) \in A^1_\infty, \quad |f^{(r)}(z)| \le 1 \text{ for } |z| \le 1\}.$$

From ([4]) it follows that

$$\varepsilon_n(B_r) = 1/(n+1) \cdots (n-r+2).$$

From Theorem 4,

$$\rho_n(B_r) \ge r_n \left(\frac{z^{n+1}}{(n+1) \cdots (n-r+2)} \right) = 1/(n+1) \cdots (n-r+2),$$

whence $\varepsilon_n(B_r) = \rho_n(B_r)$.

Theorem 4 also holds in the multidimensional case. If we consider approximations of functions analytic in the polydisc T ($|z_i| < 1$, $i = 1, \dots, k$) and continuous in $\overline{T}$, by rational functions of the form

$$R_n(z_1, \dots, z_k) = Q_n / P_n,$$

where Q_n and P_n are polynomials in $z_1, \dots, z_k$ of degree $\le n$ (in the totality of the variables), and the polynomial P_n has no zeros in $\overline{T}$, and take as norm

$$\|f\| = \max_{|z_1| = \cdots = |z_k| = 1} |f(z_1, \dots, z_k)|,$$

then for the function

$$f_0(z) = (z_1 z_2 \cdots z_k)^{n+1}$$

we have $r_n(f_0) = \|f_0\|$.

Theorem 4′ is also generalized to the case of an annulus.

Consider the space $A_\infty^{1,r}$ of functions analytic in the annulus K ($r < |z| < 1$), continuous in $\overline{K}$, with norm

$$\|f\| = \max_{|z|=1,\ |z|=r} |f(z)|.$$

Consider approximations of functions from $A_\infty^{1,r}$ by rational functions

$$R_n(z) = (b_0 + b_1 z + \cdots + b_{nz}^n)/(a_0 + a_1 z + \cdots + a_{nz}^n),$$

where the zeros of the denominator lie outside $\overline{K}$. Then, if $f_0 \in A_\infty^{1,r}$ has the properties:

a) $f_0(z)$ is analytic in the disk $|z| < 1$ (respectively, outside the disk $|z| \le r$);

b) $|f_0(e^{i\theta})| = \|f\|$, $0 \le \theta \le 2\pi$ (respectively, $|f(re^{i\theta})| = \|f\|$);

c) $f_0(z)$ has more than n zeros in the disk $|z| < 1$ (respectively, outside the disk $|z| \le r$),

then we have $r_n(f_0) = e_n(f_0) = \|f_0\|$.

If, instead of condition a), one requires only that $f_0(z)$ be meromorphic in the corresponding domain, then c) must be strengthened and it must be required that $f_0(z)$ have, in addition, more than $p-1$ zeros in the disk $|r| < 1$. (Here p denotes the number of poles of $f_0(z)$ in the corresponding domain.) The proofs of these assertions are almost a verbatim repetition of the proof of Theorem 4′.

Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.