HALF-SPACE POTENTIALS FOR THE EQUATIONS OF MOTION OF A ROTATING FLUID

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Abstract

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HALF-SPACE POTENTIALS FOR THE EQUATIONS OF MOTION OF A ROTATING FLUID

(Presented by Academician S. L. Sobolev on 29 III 1966)

L. S. Sobolev (^{1}) constructed an explicit solution of the Cauchy problem for the system of equations describing small oscillations of a rotating fluid (^{2})

$$\partial \mathbf{v}/\partial t - [\mathbf{v} \times \vec{\omega}] + \operatorname{grad} p = \mathbf{F}, \qquad \operatorname{div} \mathbf{v} = g, \tag{1}$$

where $\mathbf{v}(x,t)$ is a vector function with components $v_{x_1}, v_{x_2}, v_{x_3}$; p is a scalar function of the variables $(x,t) = (x_1, x_2, x_3, t)$; $\vec{\omega} = \omega(0,0,1)$ is a given vector of angular velocity.

In the present work explicit formulas are obtained for the solution of two boundary-value problems in the domain $D_4 = \{x_3 \geq 0,\ 0 \leq t < \infty\}$ for the homogeneous system (1) ($\mathbf{F} \equiv 0,\ g \equiv 0$),

$$\partial \mathbf{v}/\partial t - [\mathbf{v} \times \vec{\omega}] + \operatorname{grad} p = 0, \qquad \operatorname{div} \mathbf{v} = 0 \tag{1'}$$

and for one fourth-order equation associated with system (1), and some qualitative properties of the obtained solutions are analyzed.

The first boundary-value problem for system ($1'$) is solved under the conditions

$$p\big|_{x_3=0} = \varphi(x_1, x_2, t), \qquad \mathbf{v}\big|_{t=0} = 0, \tag{2}$$

the second boundary-value problem—under the conditions

$$v_n\big|_{x_3=0} = \psi(x_1, x_2, t), \qquad \mathbf{v}\big|_{t=0} = 0. \tag{3}$$

In what follows we shall denote $x' = (x_1, x_2)$, $a' = (a_1, a_2)$. Assuming that the functions $\varphi(x', t)$ and $\psi(x', t)$ are sufficiently smooth and decrease sufficiently rapidly at infinity, using the Laplace transform with respect to t and the Fourier transform with respect to x_1 and x_2, while choosing the solution of the boundary-value problem for the ordinary differential equation that tends to 0 as $x_3 \to \infty$, we obtain the solution of the first boundary-value problem for system (1′) in the form

$$v_{x_1}(x,t) = -\frac{1}{2\pi}\int_0^t d\tau \iint_{-\infty}^{+\infty} \left\{ \frac{\partial\varphi(a',\tau)}{\partial a_1}\left[\frac{x_3}{r^3}J_0\left(\frac{\rho(t-\tau)}{r}\right) + \frac{x_3\rho(t-\tau)}{r^4}J_0'\left(\frac{\rho(t-\tau)}{r}\right)\right] + \frac{\partial\varphi(a',\tau)}{\partial a_2}\frac{x_3(t-\tau)}{r^3} \right. \tag{4}$$

where J_0 is the Bessel function of order zero, $J_0'(\xi)$ is the derivative of J_0 with respect to its argument ξ,

$$\rho^2 = \sum_{i=1}^{2}(x_i - a_i)^2, \qquad r^2 = \rho^2 + x_3^2.$$

The component v_{x_2} can be obtained from v_{x_1} by replacing $\partial\varphi/\partial a_1$ by $\partial\varphi/\partial a_2$, $\partial\varphi/\partial\alpha_2$ by $-\partial\varphi/\partial\alpha_2$ in the right-hand side of formula (4) for v_{x_1};

$$v_{x_3}(x,t) = \frac{1}{2\pi}\int_0^t d\tau \iint_{-\infty}^{+\infty} \left\{ \sum_{i=1}^{2} \frac{\partial\varphi}{\partial\alpha_i}\left[\frac{x_i-\alpha_i}{r^3}J_0\left(\frac{\rho(t-\tau)}{r}\right) - \frac{(t-\tau)(x_i-\alpha_i)x_3^2}{\rho r^4}J_0'\left(\frac{\rho(t-\tau)}{r}\right)\right]\right\} d\alpha', \tag{5}$$

$$p(x,t) = \frac{1}{2\pi}\iint_{-\infty}^{+\infty} \frac{x_3}{r^3}\varphi(\alpha',t)\,d\alpha' + \frac{1}{2\pi}\int_0^t d\tau \iint_{-\infty}^{+\infty} \varphi(\alpha',\tau)\frac{x_3}{r^3}\left[\frac{x_3^2(t-\tau)}{r^2}J_0\left(\frac{\rho(t-\tau)}{r}\right) + \frac{\rho}{r}J_0'\left(\frac{\rho(t-\tau)}{r}\right)\right] d \tag{6}$$

The solution of the second boundary-value problem, when the compatibility conditions
$\psi(\alpha',0) = 0,\ \partial\psi(\alpha',0)/\partial t = 0$ are fulfilled, has the form

$$v_{x_1}(x,t) = \frac{1}{2\pi}\iint_{-\infty}^{+\infty} \psi(\alpha',t)\frac{x_1-\alpha_1}{r^3}\,d\alpha' +$$

$$+\frac{1}{2\pi}\int_0^t d\tau \iint_{-\infty}^{+\infty} \psi(\alpha',\tau)\left\{\left[\frac{(x_1-\alpha_1)x_3^2(t-\tau)}{r^5} + \frac{x_2-\alpha_2}{r^3}\right]J_0\left(\frac{\rho(t-\tau)}{r}\right) + \right.$$

$$\left. + \left[\frac{(x_1-\alpha_1)\rho}{r^4} - \frac{(x_2-\alpha_2)x_3^2(t-\tau)}{\rho r^4}\right]J_0'\left(\frac{\rho(t-\tau)}{r}\right)\right\} d\alpha'. \tag{7}$$

The component v_{x_2} is obtained from v_{x_1} by the substitution

$$(x_1-\alpha_1)\to(x_2-\alpha_2),\qquad (x_2-\alpha_2)\to-(x_1-\alpha_1)$$

in the right-hand side of formula (7) for v_{x_1},

$$v_{x_3}(x,t)=\frac{1}{2\pi}\iint_{-\infty}^{+\infty}\frac{x_3}{r^3}\psi(\alpha',t)\,d\alpha'+$$

$$+\frac{1}{2\pi}\int_0^t d\tau\iint_{-\infty}^{+\infty}\frac{x_3}{r^3}\left[\frac{x_3^2(t-\tau)}{r^2}J_0\left(\frac{\rho(t-\tau)}{r}\right)+\frac{\rho}{r}J_0'\left(\frac{\rho(t-\tau)}{r}\right)\right]\psi(\alpha',\tau)\,d\alpha', \tag{8}$$

$$p(x,t)=\frac{1}{2\pi}\iint_{-\infty}^{+\infty}\frac{\partial\psi(\alpha',t)}{\partial t}\frac{1}{r}\,d\alpha'+$$

$$+\frac{1}{2\pi}\int_0^t d\tau\iint_{-\infty}^{+\infty}\left[\frac{1}{r}J_0\left(\frac{\rho(t-\tau)}{r}\right)+\frac{\rho^2}{r^3}J_0''\left(\frac{\rho(t-\tau)}{r}\right)\right]\psi(\alpha',\tau)\,d\alpha'. \tag{9}$$

Simultaneously with system (1), one may consider a fourth-order equation for determining the function p,

$$\partial^2\Delta p/\partial t^2+\partial^2 p/\partial x_3^2=0 \tag{10}$$

(each of the unknown functions of the homogeneous system (1′) satisfies this equation), with the initial conditions

$$p|_{t=0}=\partial p/\partial t|_{t=0}=0 \tag{11}$$

and with one of two boundary conditions: for the first boundary-value problem

$$p|_{x_3=0}=\varphi(x',t) \tag{12}$$

and for the second boundary-value problem

$$\left.\frac{\partial^3 p}{\partial t^2\partial n}+\frac{\partial p}{\partial x_3}\cos nx_3\right|_{x_3=0}=-\frac{\partial\psi(x',t)}{\partial t}-\frac{\partial^3\psi(x',t)}{\partial t^3}\equiv\psi_1(x',t). \tag{13}$$

(The boundary condition (13) is obtained as a consequence of the boundary condition $v_n|_{x_3=0}=\psi(x',t)$.)

The solution of problem (10)–(12) is given by formula (6), provided that the compatibility conditions of the initial and boundary conditions on the edge $x_3 = 0,\ t = 0$ are satisfied:

$$\left.\frac{\partial^k \varphi}{\partial t^k}\right|_{t=0} = 0, \qquad k = 0, 1.$$

The solution of problem (10), (11), (13) is given by the formula

$$p(x,t) = -\frac{1}{2\pi}\int_0^t d\tau \iint_{-\infty}^{+\infty} \left[\frac{1}{\rho}\int_0^{p(t-\tau)/r} J_0(\xi)\, d\xi\right] \psi_1(x',\tau)\, d\alpha' \tag{14}$$

or by formula (9), which is obtained from formula (14) by integration by parts with respect to τ, provided the following compatibility conditions are satisfied:

$$\left.\frac{\partial^i \psi}{\partial t^i}\right|_{t=0} = 0, \qquad i = 0, 1, 2.$$

The solutions of all boundary-value problems in the half-spaces $x_1 \geq 0$ or $x_2 \geq 0$ are expressed by analogous formulas, which can be obtained by the cyclic substitution $x_3 \to x_1$, etc., despite the fact that there is an asymmetry in the equations and boundary conditions.

We have considered in system (1) the vector $\vec{\omega} = \omega(0,0,1)$, since, if the vector $\vec{\omega}(a_1, a_2, a_3)$ is constant, then it can be reduced to the form $\vec{\omega}(0,0,b)$ by a rotation of the coordinate axes, in view of the invariance of the system with respect to such a transformation.

If in system (1) $\omega = \vec{\omega}(a_1, a_2, a_3)$, then the fourth-order equation corresponding to equation (10) will have the form

$$\frac{\partial^2 \Delta p}{\partial t^2} + \sum_{i,j=1}^{3} a_i a_j \frac{\partial^2 u}{\partial x_i \partial x_j} = 0, \tag{15}$$

and the boundary condition (13) will become

$$\left.\frac{\partial^3 p}{\partial t^2 \partial n} + \sum_{i,j=1}^{3} a_i a_j \frac{\partial p}{\partial x_i} \cos(n, x_j)\right|_{x_3=0} = \mu(x', t). \tag{16}$$

The solution of problem (15), (16), (11) will be expressed by the formula

$$p(x,t) = -\frac{1}{2\pi}\int_0^t d\tau \iint_{-\infty}^{+\infty} \mu(\alpha', \tau) \left[\frac{1}{\rho}\int_0^{\rho(t-\tau)/r} J_0(\xi)\, d\xi\right] d\alpha',$$

where all the notation is as before, except for ρ; in this formula

$$\rho^2 = [a_1(x_2 - \alpha_2) - a_2(x_1 - \alpha_1)]^2 + [a_2 x_3 - a_3(x_2 - \alpha_2)]^2 +$$

$$+[a_3(x_1-\alpha_1)-a_1x_3]^2.$$

It is shown that the constructed solution of the boundary-value problems for system $(1')$ is unique in $L_2(D_t)$, except for the function $p(x,t)$ in problem (1), (3), which is determined up to a summand depending on t. Indeed, under certain requirements on φ and ψ, one can prove that the solution belongs to L_2. On the other hand, one can obtain an energy estimate (^{4}) for the solutions of the problems under consideration, from which the uniqueness of the solution in L_2 will follow.

System (1) and equation (10) are not solved with respect to the derivatives with respect to t and do not belong to any of the classical types. They have two families of real characteristics: 1) the hyperplanes $t=\text{const}$, i.e. the characteristics of a parabolic equation; 2) cylinders with arbitrary bases in the space $x=(x_1,x_2,x_3)$ and with generators parallel to the t-axis, which are double charac-

teristics (1) and (10). The speed of propagation of disturbances here is infinite (^{3}).

The Poisson kernels for the solution of boundary-value problems in the half-space have, with respect to t, a singularity of the type of a δ-function; this explains the presence in the formulas for the solutions (6)—(9) of terms in which integration with respect to τ is absent.

As was shown in (^{1}), the solution of the Cauchy problem for the nonhomogeneous system (1) contains integrals which are to be understood in the sense of the principal value, if the singularities in the space $x=(x_1,x_2,x_3)$ are cut out in a special way, namely by the cylinder $|x_3-a_3|\leq h,\ \rho\leq\eta$, and the limiting transition with respect to h or η (i.e., $h\to 0$ or $\eta\to 0$) is made depending on which component of the velocity is to be computed.

Using the explicit formulas of S. L. Sobolev (^{1}) for the solution of the Cauchy problem for the nonhomogeneous system (1) and our formulas for the solution of half-space nonhomogeneous boundary-value problems, one can obtain boundary and interior estimates of the solutions of boundary-value problems for arbitrary domains in various function spaces, for example of the type L_p or C_α.

By analogy with system (1), in the half-space one also considers a system which differs from (1) in the last equation, i.e., when the last equation of system (1) has the form (^{3},^{4})

$$\partial p/\partial t+\operatorname{div}\mathbf{v}=g.$$

This system describes small oscillations of a rotating fluid with compressibility taken into account.

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Note: Figure translations are in progress. See original paper for figures.

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