THE BRANCHING HYPOTHESIS AND THE RELATION BETWEEN THE LOCAL WEIGHT AND THE CARDINALITY OF TOPOLOGICAL SPACES

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Abstract

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I. I. PAROVICHENKO

THE BRANCHING HYPOTHESIS AND THE RELATION BETWEEN THE LOCAL WEIGHT AND THE CARDINALITY OF TOPOLOGICAL SPACES *

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1. A topological space is called a T^α-space if in it the intersection of any system of cardinality $< \aleph_\alpha$ of open sets is an open set. If X is a topological space, then $T^\alpha X$ denotes the space on the set X whose open base is the totality of all possible intersections of systems of cardinality $< \aleph_\alpha$ of open sets of the space X. The T^α-product of spaces $\{X_\lambda \mid \lambda \in L\}$ is the space on the abstract product $\prod_\lambda X_\lambda$, whose open base is given by fixing any system of indices λ of cardinality $< \aleph_\alpha$ together with the choice of open sets in the corresponding spaces (the Tikhonov product is a special case of ours when $\alpha = 0$). A topological space is called $\aleph_\alpha$-bicompact if from every one of its open coverings one can extract a subcovering of cardinality $< \aleph_\alpha$ (cf. (1)). A system of sets is called $\aleph_\alpha$-centered if every one of its subsystems of cardinality $< \aleph_\alpha$ has nonempty intersection. A partially ordered set S is called a branching system if every one of its initial segments $S^b = \{x \mid x \in S,\ x < b\}$ is a well-ordered set; here the type of S^b is called the order of the element b, and the least of the ordinal numbers that is greater than the orders of all elements of S is called the order of the branching system S. The cardinal number $\aleph_\sigma$ is called (strongly) inaccessible if it is regular and from $\mathfrak{n} < \aleph_\sigma$ it follows that $2^{\mathfrak{n}} < \aleph_\sigma$. $\aleph_0$ is inaccessible; however, in what follows we shall assume $\aleph_\sigma$ to be uncountable and shall reserve the indicated notation for it.

In $(2, 3)$ we proved, for an inaccessible $\aleph_\sigma$, the equivalence of the following properties:

(α). Every branching system S of order ω_σ, in which the set of elements of fixed order has cardinality $< \aleph_\sigma$, contains a well-ordered subset of type ω_σ.

(β). Every linearly ordered set of cardinality $\aleph_\sigma$ contains a subset of at least one of the types ω_σ or ω_σ^*.

$(\tilde{\gamma})$. The T^σ-product of $\aleph_\sigma$ copies of the simple two-point space is $\aleph_\sigma$-bicompact.

Remark. Unfortunately, the authors of papers (4–8) were unaware of our papers (2, 3) (see, in particular, our abstract (9)), as a result of which the following were proved again: in (4), the implication $(\alpha) \to (\tilde{\gamma})$; in (5), $(\tilde{\gamma}) \to (\alpha)$ (Theorem 2.1); in (6, 7), $(\alpha) \leftrightarrow (\tilde{\beta})$ (Theorem 5 and Theorem 1 respectively). See also Chapter 4 of (8).

2. It is easy to see, passing to the Dedekind completion, that $(\tilde{\beta})$ is equivalent to the condition

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* **Proof correction note.** The content of the paper was reported at the International Congress of Mathematicians on 25 VIII 1966.

(β). Every linearly ordered bicompactum whose weight at all points is $< \aleph_\sigma$ has cardinality less than $\aleph_\sigma$.

Using the associativity of the T^σ-product, condition $(\tilde{\gamma})$ can be replaced by the equivalent condition (γ):

(γ). The T^σ-product of $\aleph_\sigma$ copies of discrete spaces of cardinality $< \aleph_\sigma$ is an $\aleph_\sigma$-bicompact space.

In ([3]), along with the two-point space, the segment $[0, 1]$ of the number line with the discrete topology was considered. It turns out (and this constitutes the main aim of the present paper) that one obtains a statement equivalent to (β) if in condition (β) the requirement of bicompactness is strongly weakened, and the requirement of orderability is altogether discarded; namely, (β) (and consequently also (α) and (γ)) is equivalent to the following assertion:

(β^+). Every $\aleph_\sigma$-bicompact Hausdorff space (in particular, every bicompactum) whose weight at all points is $< \aleph_\sigma$ has cardinality $< \aleph_\sigma$.

This, in particular, is of interest in connection with the well-known unsolved problem of P. S. Aleksandrov on the cardinality of bicompacta with the first axiom of countability (see ([10]), p. 853, P. S. Aleksandrov' s note No. 6). In fact, the following is obtained.

Corollary. *The existence of bicompacta with the first axiom of countability and of uncountable cardinality is incompatible with the existence of strongly inaccessible numbers with property* (α)*.*

3. Theorem 1. *If* $\aleph_\sigma$ *has property* (α) *and the space* X *is an* $\aleph_\sigma$*-bicompact space of weight* $\aleph_\sigma$*, then* $T^\sigma X$ *is also an* $\aleph_\sigma$*-bicompact space of weight* $\aleph_\sigma$*.*

Proof. Let $g = \{G_\lambda\}$ be an open base of X of cardinality $\aleph_\sigma$. Then $h = \{H_\mu\}$, consisting of all possible intersections of subsystems of h of cardinality $< \aleph_\sigma$, forms an open base of $T^\sigma X$ of cardinality

$$\sum_{\alpha<\sigma} \aleph_\sigma^{\aleph_\alpha} = \aleph_\sigma$$

(see, for example, ([11]), p. 235, 22, b). It is therefore enough for us to show that $T^\sigma X$ is $[\aleph_\sigma, \aleph_\sigma]$-compact, i.e., that every $\aleph_\sigma$-centered system of closed sets of cardinality $\aleph_\sigma$ of the space $T^\sigma X$ has a nonempty intersection. Let such a system be $\{\Psi_\nu\}$; then each Ψ_ν is the intersection of some system $\{\Phi_{\mu_\nu}\}$ of cardinality $\leq \aleph_\sigma$ of closed sets $\Phi_{\mu_\nu} = CH_{\mu_\nu}$, and, by the definition of H_μ, each Φ_{μ_ν} is the union of a system of cardinality $< \aleph_\sigma$ of closed sets $F_\lambda = CG_\lambda$ of the space X. It is clear that the system of all $\Phi_{\mu_\nu} = \{\Phi_\tau\}$ is again an $\aleph_\sigma$-centered system of closed sets of the space $T^\sigma X$ of cardinality $\aleph_\sigma$, which, by associativity of the intersection operation, has the same intersection as the system $\{\Psi_\nu\}$, and it is enough for us to show that

$$\bigcap_\tau \Phi_\tau \supset \Lambda.$$

Suppose that τ runs through all ordinals $< \omega_\sigma$ and

$$\bigcup_{\lambda_\tau} F_{\lambda_\tau} = \Phi_\tau,$$

where, for fixed τ, λ_τ runs through a set of cardinality $< \aleph_\sigma$. Consider the collection S of all complexes

$$f^\eta = \{F^0, F^1, \dots, F^\tau, \dots \mid \tau < \eta < \omega_\sigma\},$$

where $F^\tau = F_{\lambda_\tau}$ and

$$\bigcap_{\tau<\eta} F^\tau \supset \Lambda;$$

we partially order these complexes by the rule

$$f^{\eta_1} < f^{\eta_2}$$

if $\eta_1 < \eta_2$ and f^{η_2} extends f^{η_1}. Since for fixed τ the number of all F_{λ_τ} is less than $\aleph_\sigma$, the number of complexes of fixed type is less than $\aleph_\sigma$. Since $\{\Phi_\tau\}$ is $\aleph_\sigma$-centered, for any $\eta < \omega_\sigma$ we have

$$\bigcap_{\tau<\eta} \Phi_\tau \supset \Lambda,$$

whence follows the existence of a complex

$$\{F^\tau \mid \tau < \eta < \omega_\sigma\}$$

of any type $\eta < \omega_\sigma$. Therefore our system S is a branching system of order ω_σ under the conditions (α), and, consequently, there exists a sequence

$$\{F^\tau \mid \tau < \omega_\sigma\},$$

every initial segment of which

$$\{F^\tau \mid \tau < \eta < \omega_\sigma\}$$

belongs to S and therefore has a nonempty intersection. It follows that the system

$$\{F^\tau \mid \tau < \omega_\sigma\}$$

is $\aleph_\sigma$-centered and consists of closed-

⋯sets of the $\aleph_\sigma$-bicompact space X, whence

$$\bigcap_{\tau<\omega_\sigma} \Phi_\tau \supset \times \bigcap_{\tau<\omega_\sigma} F^\tau \supset \Lambda,$$

which was required to be proved.

Remark. Theorem 1 implies $(\tilde{\gamma})$, since the generalized Cantor discontinuum D_σ is the bicompactum of weight $\aleph_\sigma$, and $T^\sigma D_\sigma$ is the T^σ-product of $\aleph_\sigma$ simple doublets. Since $(\gamma) \to (\alpha)$, the assertion of Theorem 1 is even equivalent to (α).

Theorem 2. *If a Hausdorff space X has weight $< \aleph_\sigma$ at all points and contains everywhere a dense subset X_0 of cardinality $\aleph_\sigma$, then the cardinality and the integral weight of X are equal to $\aleph_\sigma$.*

Proof. Let $f(m)$ be a net in X_0, defined on the directed quasiordered set $M = \{m\}$ (cf. (12)); by the **cardinality** of a net we shall mean card M. For nets $f(m)$ and $g(n)$, $n \in N$, of the same cardinality in X_0, introduce an equivalence relation by the rule: $f(m) \sim g(n)$ if there exists a similarity mapping $\varphi : M$ onto N such that $g(\varphi(m)) = f(m)$. For a fixed $\aleph_\alpha < \aleph_\sigma$ consider the class of all nets in X_0 of cardinality $\aleph_\alpha$. Since the largest number of pairwise similar directed quasiorders on sets of cardinality $\aleph_\alpha$ is $2\aleph_\alpha$, the number of all equivalence classes for nets in X_0 of cardinality $\aleph_\alpha$ will be $\aleph_\sigma \aleph_\alpha 2\aleph_\alpha = \aleph_\sigma$. Now assign to each point x, whose weight is $\aleph_\alpha$, the class of nets in X equivalent to the net $f^x(U)$, $f^x(U) \in U(x) \cap X_0$, where $\{U(x)\} = \mathfrak{U}(x)$ is a fixed base of the space X at x of cardinality $\aleph_\alpha$, ordered by the relation $\supset$. Obviously the net $f^x(U)$ converges to x. Since X is a Hausdorff space, different points correspond to different equivalence classes, for otherwise one net would converge to different points, which is impossible $((12), p.67)$. Thus the number of all points of X whose weight is $\aleph_\alpha$ is not greater than $\aleph_\sigma$, and consequently all limit points of X will be not more than $\aleph_\sigma \operatorname{card} \sigma = \aleph_\sigma$. Since all isolated points of X lie in X_0 of cardinality $\aleph_\sigma$, it follows that card $X = \aleph_\sigma$. Combining the bases $\mathfrak{U}(x)$ of all points X, we obtain an integral base of cardinality $\leq \aleph_\sigma$, so that it remains only to prove that the weight of X cannot be $< \aleph_\sigma$. But it is easy to see that every T_1-space of integral weight $\aleph_\alpha < \aleph_\sigma$ has cardinality $< \aleph_\sigma$, since in it distinct points correspond to distinct subfamilies of the integral base consisting of all basic neighborhoods of these points, and the cardinality of the space is $\leq 2\aleph_\alpha < \aleph_\sigma$.

We now prove our implication $(\alpha) \to (\beta^+)$, from which will follow the equivalence of (β^+) with the properties (α), (β), and (γ), since $(\beta^+) \to (\beta)$ is obvious. Suppose the contrary, i.e., that (α) holds, while (β^+) does not. Then, in particular, there exists an $\aleph_\sigma$-bicompact T_2-space Y, whose weight at all points is

$< \aleph_\sigma$ and $\operatorname{card} Y \geq \aleph_\sigma$. Take a subset X_0 of Y of cardinality $\aleph_\sigma$. Its closure $(Y)[X_0] = X$, by Theorem 2, has weight and cardinality $\aleph_\sigma$ and is $\aleph_\sigma$-bicompact as a closed set of the space Y. By Theorem 1, $T^\sigma X$ is $\aleph_\sigma$-bicompact, and since the weight at all points of X is less than $\aleph_\sigma$, all points of $T^\sigma X$ are isolated. Consequently, $T^\sigma X$ has a disjoint cover of cardinality $\aleph_\sigma$ by all one-point sets, from which no proper subcover can be selected at all, which contradicts the $\aleph_\sigma$-bicompactness of $T^\sigma X$.

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Note: Figure translations are in progress. See original paper for figures.