A problem on a pairing of equations of parabolic and hyperbolic types when time derivatives occur in the boundary conditions

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Abstract

We consider the problem of finding a solution to the following system of equations:

$$\frac{\partial u^{(1)}}{\partial t} = \sum_{l=0}^{2} C_{0l}^{(1)}(x) \frac{\partial^{l u^{(1)}}}{\partial x^l} + f^{(1)}(x,t), \quad x \in (a_1, b_1), \quad t \in (0,T),$$

$$\frac{\partial^2 u^{(2)}}{\partial t^2} = \sum_{\substack{k+l \le 2 \\ k \le 1}} C_{kl}^{(2)}(x) \frac{\partial^{l+k} u^{(2)}}{\partial t^k \partial x^l} + f^{(2)}(x,t), \quad x \in (a_2, b_2), \quad t \in (0,T),$$

satisfying the boundary conditions:

$$\sum_{i=1}^{2} \sum_{l=0}^{1} \left\{ \alpha_{sl}^{(i)} \left(\frac{\partial}{\partial t} \right) \frac{\partial^l u^{(1)}}{\partial x^l} \bigg|_{x=a_i} + \beta_{sl}^{(i)} \left(\frac{\partial}{\partial t} \right) \frac{\partial^l u^{(1)}}{\partial x^l} \bigg|_{x=b_i} \right\} = \gamma_s \quad (s = 1,2,3,4)$$

and the initial conditions:

$$u^{(i)}(x,0) = \Phi_0^{(i)}(x), \quad x \in (a_i, b_i) \quad (i = 1,2),$$

$$\frac{\partial u^{(2)}}{\partial t} \bigg|_{t=0} = \Phi_1^{(2)}(x), \quad x \in (a_2, b_2),$$

where (a_i, b_i) are mutually disjoint intervals sharing common endpoints, and

$$\alpha_{sl}^{(i)}(z) = \sum_{k=0}^{i} \alpha_{slk}^{(i)}(z^k), \quad \beta_{sl}^{(i)}(z) = \sum_{k=0}^{i} \beta_{slk}^{(i)}(z^k) \quad (i = 1,2),$$

while $\alpha_{slk}^{(i)}$, $\beta_{slk}^{(i)}$, and γ_s are constant values.

Under certain restrictions on the problem data, the existence of a solution is proven using methods developed by M. L. Rasulov. Under stronger restrictions, the uniqueness of the obtained solution is established.

Figures: 1. References: 4.

Full Text

Preamble

This work, published in 1967 (T. M. III, No. 6, K 517. 946.9), addresses the boundary value problems for partial differential equations of the form:

$$\frac{\partial u^{(i)}}{\partial t} = \mathcal{P}_i(x)\frac{\partial^2 u^{(i)}}{\partial x^2} + f^{(i)}(x,t)$$

where $x \in [a_i, b_i]$ and $t \in [0, T]$ for $i = 1, 2$. The system is subject to boundary conditions at $x = a_i$ and $x = b_i$ denoted by $s = 1, 2, 3, 4$, and initial conditions $u^{(i)}(x, 0) = \Phi^{(i)}(x)$. We assume the coefficients satisfy $\mathcal{P}_i(x) > 0$ and possess sufficient smoothness on the respective intervals $[a_i, b_i]$.

1. Construction of the Solution using Laplace Transforms

The solution to the system (1)-(4) is sought using the method of integral transforms. We define the spectral problem associated with the operator $\mathcal{L}$ and consider the complex parameter λ. Let $y^{(i)}(x, \Phi, \lambda)$ be the solutions to the corresponding homogeneous and non-homogeneous equations. We define the Green' s functions $G^{(i,j)}(x, \xi, \lambda)$ which satisfy the boundary conditions (6)-(8).

The formal solution can be represented via the inversion formula:

$$u^{(i)}(x,t) = \frac{1}{2\pi\sqrt{-1}}\int_\Gamma e^{\lambda t} A^{(i)}(x,\lambda) d\lambda$$

where Γ is an appropriate contour in the complex plane. As $\nu \to \infty$, we analyze the asymptotic behavior of the integrals along the arcs of the contour. Under the condition $\mathrm{Re}(\lambda) = \gamma$, we ensure the convergence of the series and the stability of the numerical reconstruction.

For $m = 0, 1, 2$, we consider the derivatives:

$$\lim_{\nu\to\infty} \frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_\nu} \frac{d^m A^{(i)}(x,\lambda)}{dx^m} e^{\lambda t} d\lambda$$

The analysis shows that for $t > 0$, these integrals converge to the derivatives of the solution $u^{(i)}(x, t)$. The boundary conditions at $x = a_i$ and $x = b_i$ are satisfied in the limit as x approaches the boundaries. Specifically, the jump conditions and the matching conditions at the interfaces are preserved by the integral representation.

2. Convergence and Regularity

We establish the existence and uniqueness of the solution in the specified functional spaces. Let $\Phi^{(i)}(x)$ be the initial data. We demonstrate that:

$$\lim_{t \to 0} u^{(i)}(x,t) = \Phi^{(i)}(x)$$

The proof relies on the properties of the Green' s function and the asymptotic expansion of the spectral parameter λ. For large $|\lambda|$, the Green' s function $G^{(i,j)}$ behaves as $O(|\lambda|^{-1/2})$, which ensures the convergence of the primary integral.

Furthermore, we provide estimates for the solution:

$$|u^{(i)}(x,t)| \leq C \max |Y_s|$$

where C is a constant depending on the domain and the coefficients of the differential operator. The regularity of the solution $u^{(i)}(x,t)$ is determined by the smoothness of the source terms $f^{(i)}(x,t)$ and the initial functions $\Phi^{(i)}(x)$. If $f^{(i)}$ and $\Phi^{(i)}$ satisfy the compatibility conditions of order k, the solution possesses continuous derivatives up to the corresponding order.

3. Final Remarks and Integral Representations

The final form of the solution for the coupled system is given by the sum of the initial state contribution and the non-homogeneous source contribution:

$$u^{(i)}(x,t) = \int_{a_i}^{b_i} G^{(i,i)}(x,\xi,t)\Phi^{(i)}(\xi)d\xi + \int_0^t \int_{a_i}^{b_i} G^{(i,i)}(x,\xi,t-\tau)f^{(i)}(\xi,\tau)d\xi d\tau$$

This representation allows for the direct calculation of the field variables across the interface of the two media. The results obtained are consistent with the general theory of parabolic equations and extend the operational method to multi-domain problems with complex boundary couplings.

References

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Note: Figure translations are in progress. See original paper for figures.