ON THE BEST UNIFORM APPROXIMATION OF CONTINUOUS FUNCTIONS BY ENTIRE FUNCTIONS OF FINITE DEGREE ON THE WHOLE REAL AXIS

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 517.512.6

MATHEMATICS

A. F. TIMAN

ON THE BEST UNIFORM APPROXIMATION OF CONTINUOUS FUNCTIONS BY ENTIRE FUNCTIONS OF FINITE DEGREE ON THE WHOLE REAL AXIS

(Presented by Academician S. N. Bernstein on 21 II 1967)

Let $F(x)$ be an arbitrary real function uniformly continuous on the whole real axis and

$$|F(x) - F(x+h)| \leq \Omega(|h|) \qquad (-\infty < x < \infty), \tag{1}$$

where $\Omega(t)$ is some modulus of continuity, i.e. a nondecreasing continuous and subadditive function defined on the half-axis $0 \leq t < \infty$, such that $\Omega(0) = 0$ (see ([5]), Sec. 3.2). For any $\sigma > 0$ denote, as usual, by

$$A_\sigma(F) = \inf_{G_\sigma} \sup_{-\infty<x<\infty} |F(x) - G_\sigma(x)| \tag{2}$$

the best uniform approximation of the function $F(x)$ by all possible entire functions $G_\sigma(x)$ of degree $\leq \sigma$ (see, for example, ([5]), Sec. 2.6.2).

In the paper ([9]) (see also ([8])) the author established a constructive duality principle in an inverse problem of the theory of best uniform approximations on a half-axis, connected with properties of the nonlinear operator of the form

$$\sup_x \{f(x) - \omega(x)y\} = \varphi(y), \tag{3}$$

which assigns to each function $f(x)$, defined on the half-axis $x \geq 0$, a certain convex function $\varphi(y)$ on the half-axis $y \geq 0$, and of the operator dual to it,

$$\inf_x \{f(x) + \omega(x)y\} = \psi(y). \tag{4}$$

Here we shall give one more application of the properties of this operator, now considered in the class of functions $f(x)$ defined on the whole axis $-\infty < x < \infty$, to the proof of the following assertion about the best approximations (2).

Theorem. Whatever the real function $F(x)$ satisfying condition (1), there exists an unbounded set of values $\sigma > 0$, depending only on the modulus $\Omega(t)$, for which the inequality

$$A_{\sigma-0}(F) \le \tfrac{1}{2}\Omega(\pi/\sigma), \tag{5}$$

holds; it is valid on the whole half-axis $\sigma > 0$ if the modulus of continuity $\Omega(t)$ is convex. Without the assumption of convexity of $\Omega(t)$, for those values $\sigma > 0$ for which inequality (5) does not hold, it may be replaced by the inequality

$$A_\sigma(F) \le \Omega(\pi/\sigma).$$

The theorem stated refines the result of S. N. Bernstein belonging here (see ([1]), p. 373, inequality (5), and also ([5]), p. 272, inequality (9)), and the estimate (5) contained in it, as is easy to see, cannot be improved. For the case when $\Omega(t) = t^\alpha$ $(0 < \alpha \le 1)$, by virtue of the well-known limit theorem of S. N. Bernstein ([2,3]), inequality (5)

can be obtained from the estimate (6)

$$E_n^*(F) \le \frac{1}{2}[\pi/(n+1)]^\alpha$$

of the best uniform approximation by trigonometric polynomials of order $\le n$ of periodic functions $F(x)$ with period 2π satisfying a Lipschitz condition of degree α with constant one, although those propositions about polygonal lines on which the derivation in (6) rests (see (6), pp. 749-750)* are not directly applicable for estimating $A_\sigma(F)$.

When passing to other, even convex, moduli of continuity $\Omega(t)$, one can no longer in an analogous way reduce inequality (5) to the corresponding estimate for periodic functions, since in the arguments in the proof of the mentioned limiting theorem of S. N. Bernstein an essential role is played by the power character of the modulus $\Omega(t) = t^\alpha$ (see (5), Sec. 5.6). However, we wish to show here that this inequality in the general case follows directly from the obvious properties, on the whole axis $-\infty < x < \infty$, of the nonlinear operator (3) for $\omega(x) = |x|$, and from one old result of M. G. Krein (4).

Indeed, substituting in (3) $f(x) = F_t(x)$, where $F_t(x) = F(x+t)$, and $\omega(x) = |x|$, denote by $R_y(t)$ the value of the function (3) and put $S_y(t) = R_y(t) + C$. Since, for any value of the arbitrary constant C and $y > 0$, the function $S_y(t)$ satisfies on the whole axis a Lipschitz condition of the first degree with constant y,** by the well-known theorem of M. G. Krein (4) (see (5), Sec. 5.6), for all $p > 0$ the inequality

$$A_p(F) \leq \sup_{-\infty<t<\infty} |F(t) - S_y(t)| + \pi y/2p \quad (y > 0). \tag{6}$$

If $F(x)$ satisfies condition (1), then for the difference $R_y(t) - F(t)$*** we have

$$0 \leq R_y(t) - F(t) \leq \sup_{x\geq 0}\{\Omega(x) - xy\} \quad (y > 0).$$

Hence it is clear that, in the present case, the majorant contained here of the first term on the right-hand side of (6) will be smallest when

$$C = -\frac{1}{2}\sup_{x\geq 0}\{\Omega(x) - xy\} \quad (y > 0). \tag{7}$$

After this, the usual minimization of the right-hand side of (6) with respect to $y > 0$, under the convexity condition on $\Omega(t)$, gives the second and, by virtue of well-known properties of the modulus of continuity (see, for example, (10), p. 82), the third assertion of the theorem. Thus, for example, for $y = y_n = n\Omega(1/n)$ $(n = 1, 2, ...)$, by the same properties of the modulus of continuity, the upper bound in (7) is always attained at some point x_n satisfying $0 < x_n < 1/n$, and it follows from (6) that, whatever the modulus of continuity $\Omega(t)$ may be, inequality (5) holds already for an unbounded sequence of values $\sigma = \sigma_n = \pi/x_n$ $(n = 1, 2, ...)$.

---

* A paper (11) was subsequently devoted to the proof of one of these propositions (see (6), p. 750).

** Obviously, when considering nonlinear operators more general than (3) and (4), of the form $\sup_x\{f(x)-\Phi(x,y)\}$ (see (9), § 1) or $\inf_x\{f(x)+\Phi(x,y)\}$, which occur also in a number of other questions connected with the regularization process (for example, dynamic programming, quasi-analytic classes, the theory of convex functions), for the modulus of continuity $\Omega(R_y; h)$ of the function

$$R_y(t) = \sup_{-\infty<x<\infty}\{F_t(x) - \Phi(x, y)\}$$

or

$$\inf_{-\infty<x<\infty}\{F_t(x) + \Phi(x, y)\}$$

the condition is always fulfilled

$$\Omega(R_y; h) \leq \sup_{|x_1 - x_2| \leq h} |\Phi(x_1, y) - \Phi(x_2, y)|.$$

*** In the author' s paper (9), the difference

$$R_y(t) - F(t) = \sup_x \{F(x+t) - F(t) - |x|y\} = \sup_x \{F(x) - F(t) - y|x-t|\}$$

as applied to the positive half-axis was considered at the point $t = 0$ for $y = M$ (see (9), § 2, p. 517, the upper bound (2.3) for the case when $\omega(x) = |x|$).

In the case where $F(x)$ is a periodic function of period 2π, the second and third assertions of the theorem reduce (see ([5]), § 2.6.21) to the known estimate (see ([7]), § 6) for its best uniform approximation by trigonometric polynomials. However, also in the periodic case such an estimate, like estimate (5), by virtue of the well-known result of Favard—N. I. Akhiezer—M. G. Krein (see ([5]), § 5.5.1), is an immediate and simple consequence of applying the aforementioned properties of the nonlinear operator under consideration, and follows independently from it on the same basis. It is clear, moreover, that in order to obtain it, instead of operator (3) one could also use the nonlinear operator (4) dual to it, with $\omega(x) = |x|$.

Dnepropetrovsk
Chemical-Technological Institute

Received
18 II 1967

REFERENCES

[1] S. N. Bernstein, *Collected Works*, 2, 84, Publishing House of the Academy of Sciences of the USSR, 1954, p. 371.
[2] S. N. Bernstein, *Collected Works*, 2, 86, Publishing House of the Academy of Sciences of the USSR, 1954, p. 399.
[3] S. N. Bernstein, *Collected Works*, 2, 90, Publishing House of the Academy of Sciences of the USSR, 1954, p. 413.
[4] M. G. Krein, DAN, 18, 615 (1938).
[5] A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, 1960.
[6] N. P. Korneichuk, DAN, 140, No. 4, 748 (1961).
[7] A. F. Timan, DAN, 140, No. 2, 307 (1961).
[8] A. F. Timan, DAN, 149, No. 5, 1041 (1963).
[9] A. F. Timan, *Izv. Acad. Sci. USSR, Ser. Math.*, 28, No. 3, 515 (1964).
[10] A. F. Timan, UMN, 20, 2 (122), 53 (1965).
[11] N. P. Korneichuk, in: *Problems of Mathematical Physics and the Theory of Functions*, vol. 1, Kiev, 1964, p. 65.