ON BANACH STRUCTURES OF CALDERÓN

MATHEMATICS

1967

SovietRxiv

Abstract

Full Text

UDC 513.88

MATHEMATICS

G. Ya. LOZANOVSKII

ON BANACH STRUCTURES OF CALDERÓN

(Presented by Academician V. I. Smirnov on 11 IV 1966)

In the present paper we consider conjugate and dual spaces to certain Banach structures introduced by Calderón ([2]). In doing so we shall apply Calderón' s construction not to structures of measurable functions, but to somewhat broader classes of semiordered spaces. We shall adhere to the terminology and notation from the theory of semiordered spaces adopted in the monograph ([1]).

Let S be an arbitrary extended K-space with unit 1; let X_1 and X_2 be fundamentals in S which are (b)-complete KN-spaces; let s be a real number, with $0 < s < 1$. Denote by X the set of all such $w \in S$ that

$$|w| \leq \lambda |u|^{1-s} |v|^s \tag{1}$$

for some number $\lambda > 0$ and some $u \in X_1,\ v \in X_2$, where $\|u\|_{X_1} \leq 1$ and $\|v\|_{X_2} \leq 1$. By $\|w\|_X$ denote the infimum of all possible λ in inequality (1). Then (cf. ([2])) $(X, \|\ \|_X)$ is a fundamental in S which is a (b)-complete KN-space. Following ([2]), we shall denote this space by $X_1^{1-s} X_2^s$. We note that the space $X_1^{1-s} X_2^s$ is completely determined by S, X_1, and X_2 and does not depend on the choice of the unit in S.

Let now $(L, \|\ \|_L)$ be a fundamental in S which is a KB-space with additive norm, and let J be the linear functional on L acting according to the formula

$$J(x) = \|x_+\|_L - \|x_-\|_L, \qquad x \in L. \tag{2}$$

If Z is any fundamental in S, we put

$$Z' = \{v : \ v \in S,\ vz \in L \text{ for every } z \in Z\}. \tag{3}$$

It is clear that Z' is naturally identified with the space $\bar{Z}$, conjugate to Z in the sense of Nakano, if to each $v \in Z'$ one assigns the functional $f_v \in \bar{Z}$, acting according to the formula

$$f_v(z) = J(vz), \qquad z \in Z. \tag{4}$$

Theorem 1*. The space $(X_1^{1-s}X_2^s)'$ is a fundamental in S, and

$$(X_1^{1-s}X_2^s)' = (X_1')^{1-s}(X_2')^s. \tag{5}$$

This theorem is a generalization of Theorem 4 of the author' s paper ([4]).

We give a plan of the proof of Theorem 1. It is easily verified that the right-hand side of equality (5) is contained in the left-hand side. To establish the reverse inclusion we prove successively:

1) If the net $0 \le x_\alpha$ $(\alpha \in A)$ converges weakly to zero in X_1, and the net $0 \le y_\alpha$ $(\alpha \in A)$ converges weakly to zero in X_2, then the net–

* In ([2]) this result is established only under the assumption that one of the spaces X_1, X_2 is reflexive.

the sequence $z_\alpha = x_\alpha^{1-s}y_\alpha^s$ converges weakly to zero in the space $X = X_1^{1-s}X_2^s$. This is proved by contradiction, using the theorem on the coincidence of weak and strong closures for a convex set in a normed space.

2) Let now $w \in (X_1^{1-s}X_2^s)'_+$. Then there exist positive linear functionals f_1 on X_1 and f_2 on X_2 such that for all $x \in (X_1)_+$ and $y \in (X_2)_+$ one has

$$J(wx^{1-s}y^s) \le [f_1(x)]^{1-s}[f_2(y)]^s. \tag{6}$$

3) Let φ_1 and φ_2 be the fully linear components of the functionals f_1 and f_2, respectively. Then for the indicated x and y,

$$J(wx^{1-s}y^s) \le [\varphi_1(x)]^{1-s}[\varphi_2(y)]^s. \tag{7}$$

4) Let $u \in X_1'$ and $v \in X_2'$ be those elements which, in formula (4), correspond to the functionals φ_1 and φ_2, respectively. Then

$$J(wx^{1-s}y^s) \le [J(ux)]^{1-s}[J(vy)]^s \tag{8}$$

again for all $x \in (X_1)_+$ and $y \in (X_2)_+$.

5) From (8) we derive that

$$w \le u^{1-s}v^s, \tag{9}$$

whence it follows that $w \in (X_1')^{1-s}(X_2')^s$.

Let us note that if in one of the spaces X_1, X_2 condition (A) is satisfied (i.e., from $x_n \downarrow 0$ it follows that $\|x_n\| \to 0$; see ([1]), p. 207), then condition (A) is also satisfied in $X = X_1^{1-s} X_2^s$. Consequently, in this case, by formula (4), one can identify $(X_1')^{1-s}(X_2')^s$ and the (b)-conjugate space of $X_1^{1-s} X_2^s$. At the same time, condition (B) (i.e., from $0 \le x_n \uparrow +\infty$ it follows that $\|x_n\| \to \infty$; see ([1]), p. 207) may be satisfied in one of the spaces X_1, X_2 and not satisfied in $X_1^{1-s} X_2^s$.

Let us note that if X_1 and X_2 are not (b)-complete KN-spaces, but merely ideals in S, then formula (5), generally speaking, no longer holds. For example, take $S = S[0,1]$, $X_1 = L^{1+0}[0,1]$, $X_2 = X_1'$, $s = 1/2$. Then

$$(X_1'^{1/2} X_2'^{1/2})' \supset L^2[0,1], \qquad L^2[0,1] \neq (X_1')^{1/2}(X_2')^{1/2} \subset L^2[0,1],$$

i.e., $(X_1'^{1/2} X_2'^{1/2})' \neq (X_1')^{1/2}(X_2')^{1/2}$.

Theorem 2. *If one of the spaces X_1, X_2 is a KB-space, and in one of the (b)-conjugate spaces X_1^*, X_2^* condition (A) is satisfied, then the space $X_1^{1-s} X_2^s$ is a (b)-reflexive KB-space.*

The proof of this theorem is based on formula (5).

Let us note that Theorem 2 is a generalization of the known criterion of Ogasawara for (b)-reflexivity in the sufficiency part (see ([1]), p. 294), which is obtained when $X_1 = X_2$ and $0 < s < 1$ is arbitrary. Theorem 2 is also a generalization of Theorem 1 of the author’ s paper ([3]) on the reflexivity of the space X_p for $p > 1$, which is obtained if for X_1 one takes an arbitrary KB-space and also sets

$$X_2 = \{x : \ x \in S, \ |x| \le \lambda 1 \text{ for some } \lambda > 0\}$$

and, for $x \in X_2$,

$$\|x\|_{X_2} = \inf\{\lambda : \ \lambda > 0, \ |x| \le \lambda 1\},$$

i.e., for X_2 one takes the KN-space of elements bounded relative to the unit element 1. One must also take $s = 1 - 1/p$.

Let us also note that a case is possible in which condition (A) is satisfied in X_1, condition (B) is satisfied in X_2, condition (A) is satisfied in one of the spaces X_1^*, X_2^*, but the space $X_1^{1-s} X_2^s$ is not only not (b)-ref-

it is reflexive, but is not even a KB-space. This will be the case, for example, if for S one takes the space of all real numerical sequences, $X_1 = c_0$, $X_2 = m$, since then for every $0 < s < 1$ one has $X_1^{1-s} X_2^s = c_0$.

In what follows $S, 1, L, J$ will have the former meaning, and by X we shall denote an arbitrary (b)-complete KN-space which is a foundation in S. We emphasize that we require no additional compatibility of the order and topology in X. As before, $0 < s < 1$.

Theorem 3. *The space* $X^{1-s}(X')^s$ *is a* (b)*-reflexive* KB*-space; moreover, for* $s = 1/2$ *it is isomorphic to a Hilbert space and*

$$X^{1/2}(X')^{1/2} = \{x:\ x \in S,\ x^2 \in L\}. \tag{10}$$

We omit the proof of Theorem 3.

We now introduce on X' the norm defined by

$$\|y\|_{X'} = \sup_{\substack{x \in X \\ \|x\|_X \le 1}} J(xy), \qquad y \in X', \tag{11}$$

i.e., the norm induced by the norm of the space X^*.

Theorem 4. *There exists a constant* $C > 0$ *such that every* $x \in L$ *can be represented in the form*

$$x = x_1 x_2,$$

where $x_1 \in X,\ x_2 \in X'$ *and*

$$\|x_1\|_X \|x_2\|_{X'} \le C\|x\|_L. \tag{12}$$

The proof of Theorem 4 is based on the use of formula (10).

Remark 1. We note that E. M. Semenov proved (see (5)) that every function $x(t)$ summable on $[0,1]$ can be represented in the form $x(t) = x_1(t)x_2(t)$, where $x_1(t) \in \Lambda(\alpha)$, $x_2(t) \in M(\alpha)$, and

$$\|x_1\|_{\Lambda(\alpha)} \|x_2\|_{M(\alpha)} \le \frac{\pi(1-\alpha)}{\sin \pi\alpha} \|x\|_L.$$

Here $\Lambda(\alpha)$ and $M(\alpha)$ are Lorentz spaces. Since $M(\alpha)$ is conjugate to $\Lambda(\alpha)$, our Theorem 4 explains this result.

Remark 2. Denote by $C(X)$ the infimum of all possible C' s in inequality (12). It can be shown that $C(X)$ is determined only by the space $(X, \|\ \|_X)$ itself and does not depend in any way on the choice of the unit 1 in S and of $(L, \|\ \|_L)$. It is clear that always $C(X) \ge 1$. If, for example, the ordinary Lebesgue space $L^p[0,1]$, $p > 1$, is taken for X, then for it $C(X) = 1$.

Remark 3. Theorem 4 does not admit an extension to the countably normed case, even if one requires that X be a KB^*-space. For example, let $S = S[0,1]$, L be the usual one, and J the Lebesgue integral. For X take the space of all functions on $[0,1]$ summable to any degree $p \ge 1$. Then X is a KB^*-space and $X' = L^{1+0}$. It is clear that not every function $x \in L$ can be represented in the form $x = x_1 x_2$, where $x_1 \in X$, $x_2 \in X'$, since every function of this form is necessarily summable to some degree $p > 1$.

Remark 4. From Theorem 4 the following follows. Let X be a (b)-complete KN-space and a foundation in $S[0,1]$. Generally speaking, it is not necessary that $X \supset M[0,1]$ or $X \subset L[0,1]$. There exists a measurable, nonnegative, almost everywhere finite function $z(t)$ on $[0,1]$ such that

$$L[0,1] \supset X \cdot z \supset M[0,1],$$

where $Xz = \{xz : x \in X\}$.

The author expresses deep gratitude to his scientific adviser, Prof. B. Z. Vulikh, for his attention.

Received
6 IV 1966

References

1. B. Z. Vulikh, *Introduction to the Theory of Semi-Ordered Spaces*, 1961.
2. A. P. Calderón, Studia Math., **24**, No. 2 (1964).
3. G. Ya. Lozanovskii, DAN, **158**, No. 3 (1964).
4. G. Ya. Lozanovskii, DAN, **163**, No. 3 (1965).
5. E. M. Semenov, *Scales of Banach spaces connecting the spaces L_1 and L_∞*, Abstract of Candidate' s Dissertation, Voronezh, 1964.

Note: Figure translations are in progress. See original paper for figures.