Lyapunov’ s first method

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Abstract

Full Text

Preamble

DIFFERENTIAL EQUATIONS

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ON THE STABILITY OF SOLUTIONS TO A CLASS OF SECOND-ORDER DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

In this paper, we investigate the stability of the solutions to a second-order linear differential equation of the form:

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0$$

where $p(t)$ and $q(t)$ are continuous functions defined for $t \geq t_0$. The problem of determining the asymptotic behavior of solutions for such equations is a fundamental task in the qualitative theory of differential equations, particularly when the coefficients are not constant.

1. Fundamental Stability Criteria

Consider the equation $\ddot{x} + p(t)\dot{x} + q(t)x = 0$. We assume that the coefficients $p(t)$ and $q(t)$ satisfy certain regularity conditions that ensure the existence and uniqueness of solutions on the interval $[t_0, \infty)$. To analyze the stability of the trivial solution $x(t) = 0$, we employ the Lyapunov function method and the comparison principle.

Suppose there exists a positive definite function $V(x, \dot{x}, t)$ such that its derivative along the trajectories of the system is non-positive. For the specific case of the equation above, we can define a candidate Lyapunov function related to the energy of the system:

$$V(x, \dot{x}, t) = \frac{1}{2}\left(\dot{x}^2 + q(t)x^2\right)$$

Taking the derivative of V with respect to t along the solutions, we obtain:

$$\dot{V} = -p(t)\dot{x}^2 + \frac{1}{2}\dot{q}(t)x^2$$

If $q(t) > 0$ and $\dot{q}(t) \leq 0$, while $p(t) \geq 0$, the stability of the system can be established under various conditions on the integrals of these coefficients.

2. Asymptotic Behavior and Bounds

The behavior of the solutions as $t \to \infty$ depends significantly on the integrability of the functions $p(t)$ and $q(t)$. As noted in [**?**], if the integral $\int_{t_0}^{\infty} p(s)ds$ diverges, the damping term plays a dominant role in the evolution of the system. Conversely, if the coefficients are periodic or quasi-periodic, the stability regions can be characterized using Floquet theory or the method of averaging.

§ 1. Lyapunov was the first to rigorously formulate the problem of the stability of motion. He provided a definition of stability that has since become classical and developed two methods for its investigation. The first method involves finding solutions in the form of series, while the second method (the direct method) is based on the use of special functions, now called Lyapunov functions.

Consider a system of differential equations of the form:

$$\dot{x} = P(t)x + f(t, x) \tag{1.1}$$

where $P(t)$ is a matrix of order n. Let us assume that $f(t, 0) = 0$, such that $x = 0$ is a solution to the system (1.1). According to Lyapunov, this solution is called stable if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|x(t)| < \epsilon$ for all $t \geq t_0$ whenever $|x(t_0)| < \delta$. If, in addition to this stability condition, it holds that:

$$\lim_{t \to \infty} x(t) = 0$$

then the solution $x = 0$ is called asymptotically stable. We shall refer to $x = 0$ as the equilibrium point (rest point). If the system has a solution $x = a$, where a is a constant vector, this point can obviously be translated to the origin $x = 0$ through a change of variables. The problem is to find conditions under which $x = 0$ is stable, asymptotically stable, or unstable.

Lyapunov proposed two methods for solving this problem. Roughly speaking, the Second Method of Lyapunov consists of the following: if we can construct a continuous function $V(t,x) = V(t, x_1, \dots, x_n) > 0$ for $x \neq 0$ such that $V(t,x) \to 0$ as $x \to 0$ uniformly with respect to t, and for $|x| > \delta > 0$ we have $V(t,x) > 0$, then the solution $x = 0$ is stable provided the derivative along the trajectories $\dot{V} = \frac{\partial V}{\partial t} + \sum \frac{\partial V}{\partial x_i}\dot{x}_i \leq 0$. If, by virtue of equations (1.1), we have $\dot{V} < 0$ and $V(t,x) \to 0$ as $x \to 0$, then the solution $x = 0$ is also asymptotically stable. For example, this occurs if $-\dot{V}(t,x) \geq W(x) > 0$ and $V(t,x) \to 0$ uniformly with respect to t as $x \to 0$. While this approach resolves the question of stability, it does not provide explicit formulas for the solutions, and beyond the fact of stability, it yields little information about the behavior of solutions starting near $x = 0$. It is true, however, that the rate of decay of the function V as $t \to \infty$ sometimes allows us to judge the rate at which $x(t) \to 0$. In this article, we will not focus on the second method, although we will occasionally note its significance in various cases.

What is the First Method of Lyapunov? This method is based on a specific constructive existence theorem for solutions of a system of differential equations:

$$\dot{x}_s = p_{s1}(t)x_1 + \dots + p_{sn}(t)x_n + \sum X_s^{(m_1,\dots,m_n)}(t)x_1^{m_1}\dots x_n^{m_n} \tag{1.4}$$

where $m_1 + \dots + m_n \geq 2$.

Erugin notes that the coefficients are continuous and bounded functions in the domain $t > t_0$. Lyapunov makes the following assumption:

$$|p_{sj}(t)| < M, \quad |X_s^{(m_1,\dots,m_n)}(t)| < A_{m_1+\dots+m_n} \tag{1.5}$$

where the functions $X_s^{(m_1,\dots,m_n)}(t)$ have a finite supremum and a non-zero infimum on any finite interval $t_0 \leq t \leq T$; that is, $|X_s^{(m_1,\dots,m_n)}(t)| < M(T)$. Then the right-hand sides of equations (1.4) converge within the interval $t_0 \leq t \leq T$. A special case of such a system occurs when the coefficients can be chosen to be independent of t; for example, when the coefficients $P_s^{(m_1,\dots,m_n)}$ are constants. Lyapunov finds the solution to such a system in the form:

$$x_s = \sum_{j=1}^{\infty} x_s^{(j)}(t, a_1, \dots, a_n) \tag{1.6}$$

where

$$x_s^{(1)} = \sum_{l=1}^{n} a_l x_{sl}(t) \tag{1.7}$$

Here, $X(t)$ is the fundamental matrix of solutions for the linear system:

$$\dot{x}_s = \sum_{l=1}^{n} p_{sl}(t)x_l \tag{1.8}$$

corresponding to the linear part of system (1.4). The terms $a_1, \dots, a_n$ are arbitrary constants, and $x_s^{(j)}(t, a_1, \dots, a_n)$ are homogeneous polynomials in $a_1, \dots, a_n$

of degree j, which serve as solutions to the corresponding non-homogeneous linear systems with initial conditions $x_i = 0$ at $t = t_0$.

Lyapunov proved that the series constructed in (1.6) converge absolutely and uniformly with respect to t on the interval $t_0 < t < T$ and for $|x_i| < g$, where g is determined via $A_k(T)$. Note also that if the matrix $A = I$, then we obtain the standard form. Lyapunov' s theorem differs from the theorems of Picard and Cauchy in its assumptions regarding the right-hand sides of the equations (1.4) and in the construction of the solution itself. The series (1.6) represent the general solution of equations (1.4) in the neighborhood of the point $t = t_0$. Furthermore, Lyapunov shows how, based on this theorem, one can construct a solution to system (1.4) such that for certain systems of a fairly general type, it remains representable in this form. Through such a construction of solutions, we obtain important information regarding the qualitative behavior of the solutions in the neighborhood of a rest point.

This analytical construction of solutions for system (1.4) is based on certain properties of the solutions of the linear system (1.8), which Lyapunov calls the first approximation. To formulate Lyapunov' s main theorem, we must present the profound classification of linear systems (1.8) introduced by Lyapunov. We shall call the real number λ the characteristic exponent (c.e.) of a function $x(t)$ if $x(t)e^{-\lambda t} \to 0$ as $t \to \infty$, while for any $\alpha > 0$, the function $x(t)e^{-(\lambda-\alpha)t}$ is unbounded. Every continuous function has a c.e. If we have a system of functions $x_1, \dots, x_n$, the smallest c.e. among these functions is called the c.e. of the system. Lyapunov proves that all (non-zero) solutions of system (1.8) have finite c.e. In total, system (1.8) has no more than n distinct c.e. If the coefficients of (1.8) are constant, then the c.e. of this system will be the real parts of the eigenvalues of the coefficient matrix $P = \|p_{ks}\|$.

It is always possible to construct a fundamental system of solutions such that the sum of their characteristic exponents $S = \lambda_1 + \dots + \lambda_n$ is minimized. Lyapunov proves that $S + \mu \geq 0$, where μ is the c.e. of the function $\exp(\int \sum p_{ii} dt)$. If $S + \mu = 0$, then system (1.8) is called regular. Lyapunov also introduced the consideration of reducible systems. Let $y_1, \dots, y_n$ be new unknown functions defined by the equalities:

$$x_i = \sum_{s=1}^{n} q_{is} y_s \quad (i = 1, \dots, n). \tag{1.9}$$

For $y = (y_1, \dots, y_n)$, we obtain the system $\dot{y} = yQ(t)$. If a transformation matrix $Q(t)$ can be specified such that it is bounded along with its inverse, and if the matrix in (1.10) is constant, then the system (1.8) is called reducible.

We shall now assume that for the entire interval $t > t_0$, system (1.8) is regular with Lyapunov characteristic exponents $\chi_1, \dots, \chi_n$. Lyapunov demonstrated that it is possible to determine functions $x_1, \dots, x_k$ formally satisfying the equations (1.4) in the form:

$$x_s = \sum a_1^{m_1} \dots a_k^{m_k} L_s^{(m_1, \dots, m_k)}(t) e^{(m_1 \lambda_1 + \dots + m_k \lambda_k)t} \tag{1.12}$$

where $a_1, \dots, a_k$ are arbitrary constants, $m_1, \dots, m_k$ are non-negative integers, and the functions L_j are independent of $a_1, \dots, a_k$, with their characteristic exponents being no greater than zero. Lyapunov proved that if the exponents $\lambda_1, \dots, \lambda_k$ in these series are negative and the absolute values of $a_1, \dots, a_k$ do not exceed a certain number $A > 0$, then these series converge absolutely and uniformly in the interval $t > t_0$. It follows from the form of these series that for sufficiently small $|a_1|, \dots, |a_k|$, we have $|x_k(t)| < \epsilon$, and $x_k(t) \to 0$ as $t \to \infty$.

If all λ_j are negative, one can set $k = n$ in (1.12), in which case the solutions represent the general solution of the equations (1.4) in the neighborhood of the origin. Thus, if the system (1.8) is regular and all its characteristic exponents are negative, then the solution of system (1.4) is asymptotically stable. This was the first proof of asymptotic stability based solely on the properties of the linear system. If we only have $\lambda_\nu < 0$ for $\nu = 1, \dots, k$, then according to (1.12), we obtain a k-parameter family of integral curves that enter the equilibrium point as $t \to \infty$. Lyapunov termed the case where the zero solution is stable —provided that the initial values satisfy certain additional relationships—as the conditional stability of the zero solution.

We also note the results of other authors concerning the particular form of system (1.4) where all coefficients in the right-hand sides are independent of t:

$$\dot{x}_s = p_{s1}x_1 + \dots + p_{sn}x_n + \sum X_s^{(m_1,\dots,m_n)} x_1^{m_1} \dots x_n^{m_n} \tag{1.13}$$

Poincaré demonstrated that system (1.13) can be transformed into the system $\dot{y}_k = \lambda_k y_k$ (1.14) via power series (1.15) that converge for sufficiently small values, provided that the real parts of all characteristic numbers of the matrix $\|p_{sj}\|$ are negative and distinct. Picard [**?**] required only that $\lambda_1, \dots, \lambda_n$ lie on one side of a line passing through the origin and that the resonance condition $\lambda_i = \sum p_j \lambda_j$ does not hold. N. N. Krasovskii was the first to point out that asymptotic stability should be distinguished from cases where solutions simply approach the origin without stability.

H. Dulac showed that if $\lambda_1, \dots, \lambda_n$ lie on one side of a line passing through the origin, then there exists a holomorphic transformation such that system (1.13) is transformed into:

$$\dot{z}_i = \lambda_i z_i + \Phi_i(z_1, \dots, z_{i-1})$$

where Φ_i are polynomials. This is an analogue of Lyapunov' s conditional stability. Subsequently, many researchers (Birkhoff, Siegel, Pliss [**?**]) solved the problem of finding transformations that linearize the system. Lyapunov termed this approach—based on the analytical construction of integral curves—the "first method." He proved the stability of the zero solution not only when the first approximation (1.8) is regular, but in certain cases even when it is not, specifically when $S + \mu > 0$ and all c.e. $\chi_k < -\sigma$. He also demonstrated that if there is at least one $\chi_i > 0$, then unconditional stability does not exist.

The first method relies on the theory of characteristic exponents developed by Lyapunov and furthered by Bogdanov, Bylov, Vinograd, Grobman, Persidsky, and Chetaev [?]. The first method not only solves the stability problem but also provides the equations of the integral curves, allowing for the study of how parameters affect the rate of approach to equilibrium. Lyapunov showed that at least an m-parameter family of integral curves enters the origin if the system has m negative characteristic exponents.

Following Lyapunov, let us consider a system of nonlinear partial differential equations:

$$\sum_{j=1}^{k}(p_{j1}x_1 + \cdots + p_{jk}x_k)\frac{\partial z_s}{\partial x_j} = q_{s1}z_1 + \cdots + q_{sm}z_m + Z_s \tag{2.1}$$

where Z_s are holomorphic functions vanishing at the origin. Suppose that the characteristic numbers $\lambda_1, \dots, \lambda_k$ of the matrix P and $\kappa_1, \dots, \kappa_m$ of the matrix Q are not connected by relations of the form $\lambda_s = \sum m_i\lambda_i + \sum p_j\kappa_j$. Lyapunov proves the existence of a unique holomorphic solution in the form:

$$x_s = \sum p_1 \dots p_k z_1^{p_1} \dots z_m^{p_m} \tag{2.2}$$

Lyapunov designated cases where the first approximation has zero characteristic numbers as "critical cases." In such instances, the behavior is determined by nonlinear terms.

Lyapunov examined the case where one characteristic root is zero while all others have negative real parts. Such a system can be rewritten as:

$$\begin{aligned} \frac{dx}{dt} &= X \\ \frac{dx_s}{dt} &= p_{s1}x_1 + \cdots + p_{sn}x_n + X_s \quad (s = 1, \dots, n) \end{aligned} \tag{2.3}$$

where X and X_s are holomorphic functions starting with terms of at least second order. From the equations:

$$p_{s1}x_1 + \cdots + p_{sn}x_n + X_s = 0 \tag{2.4}$$

we determine $x_s = \phi_s(x)$ as holomorphic functions. Substituting these into X, we obtain a function $f(x) = g_m x^m + \dots$. Lyapunov concluded: if m is even, the zero solution is unstable; if m is odd, it is unstable for $g > 0$ and asymptotically stable for $g < 0$. If $g(x) = 0$, Lyapunov finds an integral:

$$\Phi(x_1, \dots, x_n) = c \tag{2.5}$$

In this case, there exists a curve of equilibrium points, and every solution starting near the origin remains on an integral surface and approaches an equilibrium point on that surface.

Lyapunov also investigated systems with two purely imaginary eigenvalues:

$$\begin{aligned} \frac{dx}{dt} &= -\lambda y + X \\ \frac{dy}{dt} &= \lambda x + Y \\ \frac{dx_s}{dt} &= p_{s1}x_1 + \cdots + p_{sn}x_n + X_s \end{aligned} \tag{2.6}$$

He proved the existence of a family of periodic solutions:

$$x = \sum x^{(j)}c^j, \quad y = \sum y^{(j)}c^j, \quad x_s = \sum x_s^{(j)}c^j \tag{2.7}$$

In this case, there exists an integral $x^2+y^2+F = c$ (2.8). Every solution asymptotically approaches a corresponding periodic solution, proving non-asymptotic stability.

§ 3. Construction of General Solutions

Consider the system (3.1) with one zero eigenvalue. As Lyapunov demonstrated, there exists an integral $V(x_1, \dots, x_n) = c$ (3.2). By substituting the variable according to (3.2), the equations can be written as:

$$\frac{dx_s}{dt} = \sum P_{sj}(c)x_j + \dots \tag{3.4}$$

For sufficiently small c, the eigenvalues of the matrix (3.5) have negative real parts. Using the first method, we obtain the general solution $x_s = \sum A_{sj}e^{-\lambda_j t}$ ....

For the system (3.7) with two purely imaginary roots, Lyapunov uses variables r and θ (3.9). If a family of periodic solutions (3.12) exists, there is an integral surface $H = F(x, y, \dots)$ (3.13). Introducing new variables (3.14) leads to a system (3.17) where the first approximation has periodic coefficients. The fundamental solution matrix is $Z = e^{Wt}U(c,t)$ (3.19). The real parts of the characteristic exponents $\chi_i(c)$ are negative. Using the first method, we obtain the general solution $x_s = \sum a_i e^{\chi_i(c)t}\phi_i(t,c)$.

Whenever Lyapunov established asymptotic stability based on the first approximation, he obtained the general solution. If he used the second method for "doubtful cases," the general solution remained unconstructed. If the zero solution was non-asymptotically stable, Lyapunov identified an integral manifold M that is itself asymptotically stable.

§ 4. Second-Order Systems and Singular Cases

Lyapunov considered a second-order system:

$$\begin{aligned} \dot{x} &= y + X(x,y) \\ \dot{y} &= Y(x,y) \end{aligned} \tag{4.1}$$

This critical case with two zero roots has diverse topological structures. Lyapunov identified 10 different cases. In the case of a "non-classical center," he constructed the general solution using masterful transformations. In 1954, V. I. Smirnov discovered a manuscript where Lyapunov considered a system of $n+2$ equations with two zero roots. This was published in 1963. In this difficult study, Lyapunov constructed solutions for systems of the Briot-Bouquet type.

For systems with periodic coefficients, Lyapunov examined cases with one or two zero eigenvalues. In the first case, a family of periodic solutions (5.4) exists if an integral $x + F = c$ (5.5) exists. By introducing new variables (5.6), the system is transformed into (5.10), where the matrix has negative characteristic exponents, allowing for the construction of the general solution.

In the case of two zero eigenvalues with periodic coefficients (5.11), the problem remains partially unresolved. Using $x = r\cos\theta, y = r\sin\theta$ (5.12), Lyapunov seeks a formal solution (5.15). If the series converge, the zero solution is non-asymptotically stable. However, the convergence of these series remains a difficult question. For canonical systems (5.16), formal solutions always exist, but their convergence is not guaranteed.

§ 6. Canonical Systems and Integral Manifolds

Consider the canonical system (6.1) with integral $H_0 = c^2$ (6.2). Here, the series (5.22) converge, and the zero solution is non-asymptotically stable. For the system (6.4), the equilibrium point is non-asymptotically stable regardless of λ. In the transcendental case, an integral manifold appears in the neighborhood of the origin, and the stability is determined by the behavior near this manifold.

For the system (6.17) where the matrix has two purely imaginary eigenvalues, Lyapunov constructs periodic solutions (6.19). In the case of a focus, the integral curves wind as spirals. A. N. Erugin [?] constructed the equations of these spirals in the form $r = [C + (m-1)g\theta]^{-1/(m-1)}$.

In canonical systems (6.20), stability is only possible if all eigenvalues are purely imaginary. If the elementary divisors are simple, the system can be mapped to (6.22). If all λ_k have the same sign, the origin is stable. Kolmogorov [?] provided a method to prove the existence of analytic contact transformations (6.26) that preserve stability under small perturbations.

The Krylov-Bogolyubov method allows for the construction of solutions to systems like (7.11) in the form $x = a\cos(\psi+\tau) + ...$ (7.13). This method effectively isolates the principal part of the solution and accounts for non-stationary oscillations. By comparing coefficients of ϵ^k, one obtains equations for x_m (7.16-7.20). The choice of arbitrary functions in the solution allows for either convergence in a domain or better asymptotic representation. This method represents a significant development of the "first method" and has gained wide distribution in nonlinear mechanics.

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Figures

Note: 11 additional figures available online.

DIFFERENTIAL EQUATIONS

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ON THE CHOICE OF RIGHT-HAND SIDES IN THE SYSTEM OF DIFFERENTIAL EQUATIONS OF THE GRADIENT METHOD

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Introduction. The problem of finding the extremum of functions is given much attention. For its solution, various methods are attracted, which can be divided into three classes [1—3]: engineering, mathematical, and bionic. In this article, an engineering approach to the optimization problem is propos-ɾed: a system of differential equations, describing the movement of a point in phase space to an extremal point, is considered as an automatic control system, where the control action is formed as a fuuctional as function of vak fynction of $\nabla F(\mathbf{x}) = \left(\frac{\partial F(\mathbf{x})}{\partial x_1}, \frac{\partial F(\mathbf{x})}{\partial x_2}, \ldots, \frac{\partial F(\mathbf{x})}{\partial x_n}\right)$. Such an approach allows applying methods of automatic control theory.

Consider the problem of unconstrained optimization of a convex differen-ntiable function $F(\mathbf{x}) = F(\mathbf{x}_1, x_2, \ldots, x_n)$, defined on an n-dimensional Euclidean space E_n ($\mathbf{x} = (x_1, x_2, \ldots, x_n) \in E_n$, $\|\mathbf{x}\| = \sqrt{(\mathbf{x}, \mathbf{x})}$).

Let x^* be the minimum point of the function $F(\mathbf{x})$. The differential gradient method of searching for x^* and $F(\mathbf{x}^*)$ costouts in organizing movement to the point $x = x^*$ in the procapance E_n due to sanother system of differential equa-stions. In this clyae, the gradient of the fynction $F(\mathbf{x})$ is used as a measure of deviation from the point $x = x^*$. Sio, for example, in the deterministic grad-ient method, the system is most often used

$$\frac{dx}{dt} = - A \, \nabla F(\mathbf{x}), \tag{1}$$

where A is a positive-definite matrix of size $n \times n$; $\mathbf{x}$ is an n-dimensional column vector.

Obviously, the entire variety of systems of differential equations, for which $x(t) \to x^*$ as $t \to \infty$, is not limited by the system (1). Therefore, the question arises about the choice of the "optimal" system of the gradient metho.. Un-der this, one can propose the following as optimality criteria: 1) minimum convergence time to the point $x = x^*$ (i. e. entering a sufficiently small neighborhood of it); 2) min $\|x(t) - x^*\|$ for $T \leqslant T_0$, where T_0 – some given computation time, and others [4].

If we consider the function $\varepsilon(\mathbf{x}) = F(\mathbf{x}) - F(\mathbf{x}^*)$, which in the case of finding a minimum will be positivelively ($F(\mathbf{x}) \geqslant F(x^*)$), then as one of the criteria one can take the condition of monotone convergence to the point $x = x^*$:

$$\frac{d\varepsilon(\mathbf{x})}{dt} < 0 \quad \text{for} \quad x \neq x^*. \tag{2}$$

Figure 1: Figure 1

If ... $\dfrac{dx}{dt} = f(x)$, then criterion (2) will take the form

$$\frac{d\,e(x)}{dt} = (\nabla F(x), \dot{x}) = (\nabla F(x), f(x)) < 0.$$

It is advisable to use the following criterion:

$$\min \int_0^\infty e^2(x)\,dt = \min \int_0^\infty [F(x) - F(x^*)]^2\,dt.$$

Another group of practically important criteria is based on the property of 'roughness' with respect to difference approximation when calculating on a digital computer. Consider system (1) in finite differences

$$\frac{x^{k+1} - x^k}{h} = -A \nabla F(x^k)$$

or

$$x((k+1)h) = x(kh) - A \nabla F(x^k) \cdot h.$$

Expanding the vector $x((k+1)h)$ into a Taylor series

$$x(kh) + \frac{h}{1!} \cdot \dot{x}(kh) + \frac{h^2}{2!} \ddot{x}(kh) + \ldots = x(kh) - A \nabla F(kh) \cdot h,$$

where ... as ...

$$\dot{x} = -A \nabla F(x) - O(h), \tag{3}$$

where $h^{-1}(h) = \text{const}$ nph $h \to 0$.

Consequently, when solving on a digital computer, instead of system (1), we actually have system (3). The latter serves as an explanation for the fact that when solving problems by the gradient method, either the convergence process in time is very long, or oscillations arise near the point $x = x^*$. Thus, the term $O(h)$ can be interpreted as an unknown perturbation (noise) causing oscillations in system (1). Therefore, the problem of compensating for this perturbation is relevant.

1. Use of Sliding Modes. In automatic control, one of the effective methods for suppressing disturbances unknown in advance and ensuring the property of "roughness" is the introduction of a sliding mode into the system of differential equations. As is known, for the latter to exist, the right-hand side of the equations must be a discontinuous function [5]. At the point of discontinuity (i.e., on the switching surface), a sliding mode may occur.

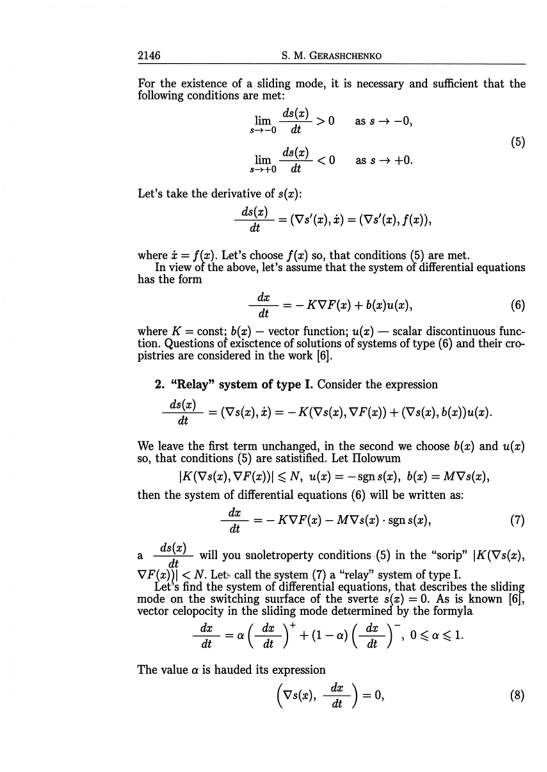
In our problem, oscillations arise near the point $x = x^*$, where $\nabla F(x^*) = 0$.

Let us consider a surface passing through the point $x = x^*$:

$$s(x) = (\mathbf{c}, \nabla F(x)) = 0, \tag{4}$$

where $\mathbf{c}$ is an arbitrary vector, its components can be not only constant, but also dependent on x. Let us organize sliding on surface (4).

Figure 2: Figure 2



For the existence of a sliding mode, it is necessary and sufficient that the following conditions are met:

$$\lim_{s \to -0} \frac{ds(x)}{dt} > 0 \quad \text{as } s \to -0,$$
$$\lim_{s \to +0} \frac{ds(x)}{dt} < 0 \quad \text{as } s \to +0. \tag{5}$$

Let's take the derivative of $s(x)$:

$$\frac{ds(x)}{dt} = (\nabla s'(x), \dot{x}) = (\nabla s'(x), f(x)),$$

where $\dot{x} = f(x)$. Let's choose $f(x)$ so, that conditions (5) are met.

In view of the above, let's assume that the system of differential equations has the form

$$\frac{dx}{dt} = -K\nabla F(x) + b(x)u(x), \tag{6}$$

where $K = \text{const}$; $b(x)$ – vector function; $u(x)$ — scalar discontinuous function. Questions of exisctence of solutions of systems of type (6) and their cropistries are considered in the work [6].

2. "Relay" system of type I. Consider the expression

$$\frac{ds(x)}{dt} = (\nabla s(x), \dot{x}) = -K(\nabla s(x), \nabla F(x)) + (\nabla s(x), b(x))u(x).$$

We leave the first term unchanged, in the second we choose $b(x)$ and $u(x)$ so, that conditions (5) are satistified. Let Πolowum

$$|K(\nabla s(x), \nabla F(x))| \leqslant N, \; u(x) = -\operatorname{sgn} s(x), \; b(x) = M\nabla s(x),$$

then the system of differential equations (6) will be written as:

$$\frac{dx}{dt} = -K\nabla F(x) - M\nabla s(x) \cdot \operatorname{sgn} s(x), \tag{7}$$

a $\frac{ds(x)}{dt}$ will you snoletroperty conditions (5) in the "sorip" $|K(\nabla s(x), \nabla F(x))| < N$. Let call the system (7) a "relay" system of type I.

Let's find the system of differential equations, that describes the sliding mode on the switching surface of the sverte $s(x) = 0$. As is known [6], vector celopocity in the sliding mode determined by the formyla

$$\frac{dx}{dt} = \alpha\left(\frac{dx}{dt}\right)^{+} + (1-\alpha)\left(\frac{dx}{dt}\right)^{-}, \; 0 \leqslant \alpha \leqslant 1.$$

The value α is hauded its expression

$$\left(\nabla s(x), \frac{dx}{dt}\right) = 0, \tag{8}$$

Figure 3: Figure 3

where

$$\left(\frac{dx}{dt}\right)^{+} = -K\nabla F(x) - M\nabla s(x) \quad \text{at } s > 0,$$
$$\left(\frac{dx}{dt}\right)^{-} = -K\nabla F(x) + M\nabla s(x) \quad \text{at } s < 0. \tag{9}$$

Determining α from relations (8) and (9), we obtain that the motion on the sliding surface $s(x) = 0$ is dencribed by the following system:

$$\frac{dx}{dt} = -K\nabla F(x) + \nabla s(x)\frac{K(\nabla s(x), \nabla F(x))}{\|\nabla s(x)\|^2}. \tag{10}$$

Calcucating $\frac{ds(x)}{dt}$ by virtue of system (10), we obtain

$$\frac{ds(x)}{dt} = -K(\nabla s(x), \nabla F(x)) + K\|\nabla s(x)\|^2\frac{(\nabla s(x), \nabla F(x))}{\|\nabla s(x)\|^2} \equiv 0.$$

Thus, $s(x) = 0$ is the first integral of the system of differential equations ot dencribing the sliding mode. Therefore, the order of system (10) can be reduced.

We will porwise the stability of solutions to sustems (7), (10) by the direct method of Lyapunowv. Note that this method was also applied earlier [7] as a method for solving nonlinerery programming problems with a single extremal point.

3. Hitting the sliding surface and stability theorems. The usual scheme for proving stability in systems with sliding is as follows. It is investigated under what conditions any point of the phase space hits the sliding surface $s(x) = 0$ at some moment in time, nnt excedomte tume one monent in time, and then the motion on the sliding surface is considered. Following papers [8, 9], we consider the change in $s^2(x)$, assuming M to be a large number,

$$\frac{d\left(\frac{1}{2}s^2(x)\right)}{dt} = s(x)(\nabla s(x), \dot{x}) =$$
$$= -M\|\nabla s(x)\|^2|s(x)|\left[1 + \frac{K(\nabla s(x), \nabla F(x))}{M\|\nabla s(x)\|^2}\right].$$

For the condition of hitting $\frac{d\left(\frac{1}{2}s^2(x)\right)}{dt} < 0$, it is required, that the second term in absolute value be less than equity, i. e.

$$\left|\frac{K(\nabla s(x), \nabla F(x))}{\|\nabla s(x)\|^2}\right| < M \tag{11}$$

for ever x, benlangling to the sosephocte $s(x) = 0$. Hequentity (11) gives the condition, imposed on the vector c to ensure that ckollisnating on the solephocte $s(x) = 0$.

Figure 4: Figure 4

To investigate hitting on $s(x) = 0$, we let $M \to \infty$ and introduce slow time $\tau(t = \nu\tau,\ \nu^{-1} = M)$. Then for $\nu = 0$

$$\frac{d\left(\frac{1}{2}s^2(x)\right)}{d\tau} = -|s(x)|\|\nabla s(x)\|^2 < 0.$$

Using the theorem on continuous dependence of solutions on the right-hand side, we conclude that to ensure hitting the sliding surface, it is necessary to take the number M sufficiently large ($M > M_0$).

Theorem 1. Let a twice differentiable function $F(x) = F(x_1, x_2, \ldots, x_n)$ be given and the system of equations $\nabla s(x) = a\nabla F(x)$ for $a \neq 0$ has no solutions other than $x = x^*$. Then the minimum point $x = x^*$ is an asymptotically stable in the large equilibrium position of the system (10), and the function $V(x) = F(x) - F(x^*)$ is a Lyapunov function for the system (10).

Proof. By assumption, the point $x = x^*$ is the only point where $\nabla F(x) = 0$. Calculating the total derivative with respect to time of $V(x)$, due to the system (10) we have

$$\frac{dV(x)}{dt} = (\nabla V(x),\ \dot{x}) = \frac{-K\|\nabla F(x)\|^2\|\nabla s(x)\|^2 + K(\nabla s(x),\ \nabla F(x))^2}{\|\nabla s(x)\|^2}.$$

By the Schwarz inequality

$$-\|\nabla F(x)\|^2\|\nabla s(x)\|^2 + (\nabla s(x),\ \nabla F(x))^2 \leqslant 0,$$

moreover, equality to zero is possible only in the case of linear dependence of the vectors $\nabla s(x)$ and $\nabla F(x)$. But by the condition of the theorem, the equality $\nabla s(x) = a\nabla F(x)$ is either possible at the point $x = x^*$, or impossible in general.

Thus, the function $V(x) = F(x) - F(x^*)$ satisfies the conditions of Lyarapunısov's theorem and, therefore, $\lim x(t) = x^*$ as $t \to \infty$.

Theorem 2. Let a twice differentiable function $F(x)$ be given, the point $x = x^*$ be the minimum point of $F(x)$, and the system of equations

$$\nabla F(x) + \frac{M}{K}\nabla s(x)\,\mathrm{sgn}\,s(x) = 0 \tag{12}$$

has no solutions other than $x = x^*$. Then the phase trajectory of the system (7), passing through an arbitrary point of the phase space E_n, ends at the point $x = x^*$ as $t \to \infty$. The function $V(x) = K[F(x) - F(x^*)] + M|s(x)|$ is a Lyapunov function for the system (7).

Proof. Let us take the total derivative with respect to time of the function $V(x)$:

$$\frac{dV(x)}{dt} = [K\nabla F(x) + M\nabla s(x)\cdot\mathrm{sgn}\,s(x)][-K\nabla F(x) - M\nabla s(x)\cdot\mathrm{sgn}\,s(x)],$$

then

$$\frac{dV(x)}{dt} = -(\nabla V(x),\ \nabla V(x)) \leqslant 0. \tag{13}$$

From the condition (12) of the theorem, it follows that $\frac{dV(x)}{dt} \neq 0$ for $x \neq x^*$.

Therefore, the function $V(x)$ is a Lyapunov function for the system of differential equations (7).

Figure 5: Figure 5

We will show further that here it is possible to construct also a **general solution in the neighborhood of the origin of coordinates.**

Lyapunov studied also such a system (1.4) with constant coefficients p_{kl} and $P^{(s)}_{(m)} \ldots m_n)$, in which the matrix of coefficients $P = \| p_{sl} \|$ of the linear system (1.8) has two purely imaginary roots, and the real parts of the remaining roots are negative. Such a system after simple transformation can be rewritten in the form

$$\frac{dx}{dt} = -\lambda y + X, \qquad \frac{dy}{dt} = \lambda x + Y, \tag{2.6}$$

$$\frac{dx_s}{dt} = p_{31}x_1 + \ldots + p_{3n}x_n + \alpha_s x + \beta_s y + X_s,$$

where X, Y, X_s — holomorphic functions in the neighborhood of the point $x = y = x_1 = \ldots = x_n = 0$, not containing free and linear terms, p_{kl}, α_s, β_s — real numbers and the real parts of all roots of the matrix $P = \| p_{kl} \|$, are negative.

Based on the solutions (2.2) of system (2.1) (obtained using the expansions (1.12)) Lyapunov reduces the study of system (2.9) to the study of such a system of $n+2$ equations (2.6), in which X and Y vanish together with $x = y = 0$. Further this system (2.6) is investigated in the following way. First, the question is posed about the existence of a family of solutions in the form

$$x = x^{(1)}c + x^{(2)}c^2 + \ldots, \quad y = y^{(1)}c + y^{(2)}c^2 + \ldots, \qquad x_s = x_s^{(1)}c + x_s^{(2)}c^2 + \ldots \tag{2.7}$$

Here c — arbitrary constant and the coefficients $x^{(o)}$, $y^{(s)}$, $x^{(c)}$, $x_\varepsilon^{(c)}$ are pe- periodic, with period 2π functions $\tau = \dfrac{(t-t_0)\lambda}{1 + h_2c^2 + \ldots}$, where h_2, h_3, $\ldots$ — some, subject to determination constants. If, sadisifying formally the equations (2.6), we find all coefficients of the series (2.7) periodic, teen, as shown by Lyapunov, these series (2.7) will be also convengent for sucfictently small $|c|$. This will find a family of periodic solutions (2.6).

How is the question about the stability of the zero solution in the case of the existence of a family of periodic solutions resolved? Such a question arises, since (2.7) is not a general solution in the neighborhood of the origin of coordinates. Lyapunov shows that in this case for the system (2.6) there exists an integral of the form

$$x^2 + y^2 + F(x_1, \ldots, x_n, x, y) = x^2 + y^2 + \sum_{m>1, m+m_1+\ldots+m_n>2} U_{m_1}^{(m_1 \ldots m_n)}(x, y)\, x_1^{m_1} \ldots x_n^{m_n} = c, \tag{2.8}$$

where F — holomorphic function in the neighborhood of the point $x_1 = \ldots = x_n = x = y = 0$ function. Conversely, if we have the integral (2.8), then we uneve also a samily of periodic solutions (2.7). Each of the solutions (2.7) is located on one of the integralizat coverphaces (2.8), and, conbericly, on each sunface (2.8) is located one of the periodic solutions (2.7). Further Lyapunov showed that every solution of system (2.8), not coiniading with the coorterstrying soverface (2.8) (initial values of this solution determine c in (2.8), i.e. determine the integral coterphaces araea in olistemoe (2.8)), asymptotically approaches the coorterstrying periodicologic solution (2.7).

Figure 6: Figure 6

method. But in all these cases he could have finished the solution of the problem also by the first method, by constructing the general solution in the vicinity of the perturbed solution. We will show this. It is possible to make the following remark on this.

Lyapunov in cases of the existence of integrals (2.5) and (2.8) did not find the general solution in the vicinity of the origin of coordinates of unknown functions by the first method. But it is possible to do this. We will show this in § 3.

§ 3. Consider the system of equations studied by Lyapunov:

$$\frac{dx}{dt} = X, \quad \frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n + X_s \quad (n = 1, \ldots, n), \tag{3.1}$$

where X, X_s are functions that are holomorphic in the vicinity of the point $x = x_1 = \ldots = x_n = 0$, which do not have free and linear terms in their expansions. The coefficients of these series and p_{s1} are constant. We will also assume 1) that the functions X and X_s at $x_1 = \ldots = x_n = 0$ vanish, and the real parts of all roots of the matrix $\mathbf{P} = \|p_{s1}\|$ are negative. As Lyapunov showed, in this case there exists an integral of the system

$$x = c + f(x_{11} \ \ldots, \ x_n, \ c), \tag{3.2}$$

where f is a function that is holomorphic in the vicinity of the point $x_1 = \ldots = x_n = c = 0$, which vanishes at $x_1 = \ldots = x_n = 0$. The zero solution of system (3.1) in this case is stable (non-asymptotically). It is precisely in this case that there exists a curve

$$x = x(c), \quad x_s = x_n(c) \quad (x = x_s = 0 \quad \text{npu } c = 0), \tag{3.3}$$

passing through the origin of coordinates and consisting of equilibrium points. This curve intersects each of the integral surfaces (3.2) at one point. The integral surfaces (3.2) surround the origin of coordinates and shrink to the origin of coordinates as $c \to 0$.

Each of the solutions of system (3.1), starting in a sufficiently small vicinity of the origin of coordinates, without leaving the corresponding surface (3.2), approaches the equilibrium point (3.3) on this surface. We set the problem of constructing these integral curves or constructing the form of the general solution of system (3.1) in the vicinity of the origin of coordinates. This can be obtained in the following way.

Since $X_s = 0$ for $x_1 = \ldots = x_n = 0$, then, substituting the variable x in X_s by formula (3.2), we obtain functions $\overline{x}_s(x_1, \ldots, x_n, c)$ that are holomorphic in the vicinity of the point $x_1 = \ldots = x_n = c = 0$. In other words, after such a substitution, the last n equations of (3.1) can be written in the form

$$\frac{dx_o}{dt} = q_{s1}(c)\,x_1 + \ldots + q_{sn}(c)\,x_n + \overline{X}_s(x_1, \ \ldots, \ x_n, \ c) \quad (s = 1, \ \ldots, \ n) \tag{3.4$_1$}$$

or, as Lyapunov writes,

$$\frac{dx_c}{dt} = (p_{s1} + c_{s1}(c))\,x_1 + \ldots + (p_{sn} + c_{sn}(c))\,x_n + \overline{X}_s(c, \ x_1, \ \ldots, \ x_n). \tag{3.4}$$

[1]) Lyapunov reduces the general case to this case, when there is one zero characteristic part, and the real part of the rest is <0.

Figure 7: Figure 7

Let us denote [1] these real parts by $\lambda_1(c), \ldots, \lambda_n(c)$. According to the first method, $z_1, \ldots, z_n$ from (3.17) we will obtain in the form (1.12)

$$z_s = \sum_{m_1+\ldots+m_n>1} L_s^{(m_1 \ldots m_n)}(t, c)\, \alpha_1^{m_1} \ldots \alpha_n^{m_n} e^{\sum_{l=1}^{n} m_l \lambda_l(c) t} \tag{3.21}$$

$$(s = 1, \ldots, n),$$

where the real parts of the functions $L_s^{(m_1 \ldots m_n)}(t, c)$ are not greater than zero, and $\alpha_1, \ldots, \alpha_o$ are arbistrary constants. Substituting those z_s into (3.16), we will obtain also z in the same form:

$$z = c + \sum_{m_1+\ldots+m_n>1} L^{(m_1 \ldots m_n)}(t, c)\, \alpha_1^{m_1} \ldots \alpha_n^{m_n} e^{\sum_{l=1}^{n} m_l \lambda_l(c) t} \tag{3.22}$$

In the same form we will obtain r and $x_1, , x_n$, if we substitute (3.21) and (3.22) into (3.14). And then we will obtain also x, y according to (3.9). As for θ, we will obtain it as a function of t from (3.10), and at the same time there will appear one more arbitrary constant c_1. Since in (3.10) Θ is a periodic function of θ, then $\theta = \theta(t)$ will be determined [2] for all t and can be obtained by Picard's method in the form of a series, converging for all finite values of t.

We have obtained the general solution of the system (3.7) in the vicinity of the origin of coordinates with arbitrary constants $\alpha_1, \ldots, \alpha_o$, cnd c_1. All these series will converge with the speed of an exponential function, therefore the solutions can easily be obtained approximately with the desired accuracy. Now it is possible to study the influence of various parameters, entering into the right parts of the equations (3.7), on the behavior of integral curves as $t \to \infty$.

Summarizing everything previously stated, the following can be said. Every time when Lyapunov obtained asymptotic stability of the zero solution, generated only by the property of the solution of the first approximation, he obtained also the general solution, the first approximation, he obtained also the general solution in the vicinity of $x_1 = \ldots = x_n = 0$ and for all $t > t_0$. If he obtained asymptotic stability of the zero solution, solution in a doubtful case, using the second method, then he could not obtain the general solution. (2). This has not been done and up to now. Thus, in this case we have only the fact of asymptotic stability. If the zero solution of the given system turned out to be unstable, then Lyapunov found in the vicinity of the point $x_1 = \ldots = x_n = 0$ a certain integral manifold [3], consisting of integral curves (entirely situated near the point $x_1 = \ldots = x_n = 0$, i.e. for all t), which turned out to be asymptotically stable, i.e. all solutions, starting near the point $x_1 = \ldots = x_n = 0$, asymptotically approached one of the integral curves of this manifold as $t \to \infty$.

The asymptotic stability of this manifold was proved by Lyapunov by the second method.

---

1) If we introduce new unknowns $y = (y_1, \ldots, y_n)$ into the system (3.17) using the formula $(z_1, \ldots, x_n) = (y_1, \ldots, y_o)\, U(e, \theta)$, then for y we obtain a system, the first ap-- prosimation of which will have a constant matrix of coefficients $W(e)$. Here $(y_1, \ldots, y_n)$ is a row matrix.

2) We have $\frac{dB}{dt} = R(\theta, t)$, $R(\theta + 2\pi, t) = R(\theta, t)$ and $\left| \frac{\partial R(\theta, t)}{\partial \theta} \right.$ as a function of time organved by t and periodic in θ.

3) N. D. Moiseev called this fact perevival partial integration (the third method).

Figure 8: Figure 8

$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n + X_s(x, y, x_1, \ldots, x_n) \quad (4.2)$$

$$(s = 1, \ldots, n),$$

where X, Y and X_s — holomorphic functions in the neighterhood of the origin, not containing free and linear terms[1]). This work by A. M. Lyaponov was published by Leningrad University Press in 1963 under the title "Investigation of one of the special claves of the problem of yctoibility montion". In this phenomenally difficult investigation, some caves are passmatred using the stopord method, while opyries, the most difficult, by the first method, building pemenue in the okinects of the origin. It is interected to notite, that there Lyaponov was forced to build concompenently permeiion of the cystem of yzantination of Briot and Boukeet tine:

$$x\frac{dz}{dx} = -(\alpha + 1)z + xZ, \alpha > 0 \text{ integer},$$

$$\frac{x^{\alpha+1}}{\alpha}\frac{dz_s}{dx} = p_{s1}z_1 + \ldots + p_{sn}z_n + Z_s \; (s = 1, \ldots, n),$$

where Z and Z_s — are holomorphic functions x, z, z_σ in the neighterhood of the origin. Namelly, he considered the bonpoc of the сущectrnees and napature peuenion of this systeme, vuvellling togeter with $x \to 0$. This problem in celf cabe is not ogne of the nerivex. For Lyaponov one cavas emaarsd uncomporter. This саучае insyred to V. A. Pliss [9]. Pliss was also вынужден to построить решенне custeme уравненнй методом последовательных прнблнженнй (пргнпа, эта custema вспомогательная, но блнзкая к рассмотранвемед). Здесь Llayonov вынужден был по-разному подходить к решенню задачн. Но это — следствне pasnus сломвеіх топологнческнх картнп расположення ннтегральных крнвых в окрестностн начала коордннт н разрышность analurinveckнх представленнй н топологнческнй картнны как функціon параметрєs custемы (4.2). Во многгх, конечно, Лаaynov вынужден был попутно здесь выявнть н качественную, очень ннтереспую картнну в окрестностн начала коорднnат. Напрнмер, bonpoc of сущестmaess and muliennost nepiodinческнх решенпй in ix contpoction, of yctofivillnocth stux pewenud. Pliss, in the paiis ykazated there, также строил решенне в окрестностн discrpetnoric множества перноднческнх решенпй, discoyerred Лaynonow.

Let us also otmete the folllowing. B custeme (2.3), the praser stopes are holomorphic fynction in the neightboroth of the начала. N. B. Munir [10] pacsmitred the custemy

$$\frac{dy}{dt} = Y(y, x), \; \frac{dx}{dt} = Ax + X(y, x),$$

whee A — a constant matruca, ale efiancneat sos kotopoid umeet ergutately real part, se vactu, and $Y(y, x)$, $X(y, x)$ vanait mvro at nyre por $x = y = 0$ u in the neightnooth of that towki nmeet contepuious partiani derienatives. There y, Y — ckalarps, a x, X — веkтops. Munir, npu ucclegoвaious вопроces of yctoйvilnостн решення $x = y = 0$, also arrived at a ono-parametper-

[1]) Already from statements by Lyapunov when noting the article on system (4.1) in 1883, it was clear cliaro. that this work exists. Clearlly note the great parody of V. P. Basov, editor of this station, so totally unfamiliar with Lyapunov, — and this is not an asrige sada.

Figure 9: Figure 9

integral manifold, in the neighborhood of which he constructed a general solution (by the method of V. A. Pliss). This represents a development of Lyapunov's first method.

§ 5. Now let us consider such systems, in which the right-hand sides are periodic functions of t. From such custems, related to cowntnful cases, Lyapunov pacsmotped two classes.

1. When the first apubliximation meet odne xapracteristic exphonent, pavnal to nyro, and all the others orgunıtively.

2. The first apublicimation meet two nyro xapracteristic espsonents, an acl the others orgunatively.

Pascmotper the first clyvae, korda the custema (1.4) gaden, $p_{kl}(t)$ and $P^{(s)}_{(m_1 \ldots m_n)}(t)$ — fynktion periodiveckic with period ω u nepsal apublximation (1.8) umeet odne xapracteristic espeonents, pavnal to nyro, and ace other are orgunıtively. Spec cybests linear real treafpasosation ([5], § 8)

$$y_k = \sum_{\nu=1}^{n} q_{k\nu}(t) x_\nu \quad (k = 1, \ldots, n), \tag{5.1}$$

rде matruцa

$$Q = \|q_{kl}(t)\| \tag{5.2}$$

is periodivecke with period ω, and a onterderomresounots, not iencounsting, u takun, чto the nobex neuxectins $y_1, \ldots, y_n$ molyvis a custem of brda

$$\frac{dx}{dt} = X, \quad \frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n + p_s x + X_s, \tag{5.3}$$

rde X u X_s будyt fynktion takers its same xaraktery, as b custem (1.4), a p_{kl}, p_s — realrerennыe constoants. Plus, tiom вещественные vactu bcex x. ч. matriqus $P = \|p_{kl}\|$ будут otrputateльnue.

Lyapunov nokasad, чto there moryt botperensers two clyvaes.

1. This custema imeet a family of periodiveckic pellution of brda

$$\begin{aligned} x &= c + u^{(2)} c^2 + u^{(3)} c^3 + \ldots, \\ x_s &= u_s^{(1)} c + u_s^{(2)} c^2 + \ldots, \end{aligned} \tag{5.4}$$

rde $u^{(i)}$, $u_s^{(i)}$ — nepiodiveckic with periodis ω, a c — proursolnary noctoян-mat c malem $|c|$.

2. There is takue family of periodiveckic pelienud.

Btopod clyvad is detected by a kinewive number of operations (but pe-eviously unksnown, ckoньko takux operations needd to be donates). If it is discovered, then, applying the stopod metod, Lyapunov nokavisaers, чto sdere there will either asymptotic yctohiñiñity of the nero pelienion, aldo an-urstabilinocth.

In the firsst clyvae Lyapunov dokavisaet cymectres of the integrale

$$x + F(x, x_1, \ldots, x_n, t) = c, \tag{5.5}$$

где F — holomorphic in the okpectnost tovku $x = x_1 = \ldots = x_n = 0$ fynction, kosфfficients paxpansion into a perio of which are nepiodiveckeeva periodod ω fynction of t. For kawgrd c (sucificiently small in modylus) on the inter-grale (5.5) lies the cootвercтвyvening periodiveckue pelienion (5.4). This nepiodiverckic pelienue bудet asymptotierally yctohilited in classe of thex peluenion, which havinairees (and, consequently, remains) on the inter-gralished noberxnocte (5.5). Periodivcecklc pelieniuв (5.4), obviouslo,

Figure 10: Figure 10

...contract to the origin as $c \to 0$. The zero solution of system (5.3) will be stable here, but not asymptotically. Thus, here again Lyapunov uses the first method, and we do not see another way. True, having found the integral (5.5), he again proves the asymptotic stability of the periodic solutions (5.4) on the basis of the second method, but, firstly, after the constroution of the solutions (5.4) or the integral (5.5), and, secondly, he could have finished the investigation not by the second method, but by the first.

In fact, having found the family (5.4), Lyapunov introduces new variables $z, z_1, \ldots, z_n$ by the equations

$$x = z + u^{(3)} z^2 + u^{(3)} z^3 + \ldots, \qquad x_s = z_s + u_s^{(1)} z + u_s^{(2)} z^2 + \ldots, \tag{5.6}$$

which transform the system (5.3) into the system

$$\frac{dz}{dt} = Z, \quad \frac{dz_s}{dt} = p_{s1} z_1 + \ldots + p_{sn} z_n + Z_s \quad (s = 1, \ldots, n). \tag{5.7}$$

Here Z and Z_s are of the same character as X and X_s in (5.3), but, in addition to this, Z and Z_s vanish for

$$z_1 = \ldots = z_n = 0.$$

Thus, we have the solution

$$z = c, \quad z_1 = \ldots = z_n = 0, \tag{5.8}$$

which cortresponds to the family (5.4).*

For the system (5.7) we have the integral

$$z = c + f(z_1, \ldots, z_n, c, t), \tag{5.9}$$

where f is a function holomorphic in the neighborhood of the point $z_1 = \ldots = z_n = c = 0$ and periodic with period ω with respect to t, not containing terms of the first dimension with respect to $z_1, \ldots, z_n, c$ and a free term. Now, replacing the variable z in (5.7) by the formula (5.9), we obtain the system

$$\frac{dz_s}{dt} = (p_{s1} + c_{s1}(c))\, z_1 + \ldots + (p_{sn} + c_{sn}(c))\, z_n + Z_s, \tag{5.10}$$

where Z_s vanishes for $z_1 = \ldots = z_n = 0$ and does not contain terms of the first dimension with respect to $z_1, \ldots, z_n$. Here $c_{kl}(c)$ are functions holomorphic with respect to c, vanaising at $c = 0$. Thus, for all sufficiently small $|c|$, the constant matrix

$$P = \|p_{kl} + c_{kl}(c)\|$$

must have characteristic numbers with negative real parts. According to formulas (1.12), we obtain the general solution in the neighborhood of the point $z = z_1 = \ldots z_n = 0$ or, what is the same, in the neighborhood of the periodic solutions (5.4), from which the aforementioned asymptotic stability of the solutions (5.4) will follow. We have thus obtained the general solution in the neighborhood of the point $x = x_1 = \ldots = x_n = 0$.

Figure 11: Figure 11

Now let us consider a system which, after transformation, can be written in the form

$$\frac{dx}{dt} = -\lambda y + X(x, y, x_1, \ldots, x_n, t),$$
$$\frac{dy}{dt} = \lambda x + Y(x, y, x_1, \ldots, x_n, t), \tag{5.11}$$
$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n + p_s x + q_s y + X_s,$$

where X, Y and X_s are holomorphic functions in the neighborhood of the point $x = y = x_1 = \ldots = x_n = 0$ with ω-periodic coefficients as functions of t without free and linear terms; p_{ik}, p_s and q_s are real constants and the real parts of the characteristic matrix $\|p_{kl}\|$ are negative. This is that second case with right-hand sides periodic with respect to t, which we mentioned at the beginning of this section. The question of the stability of the zero solution of this system has not been solved even now. We shall see why.

Lyapunov, setting

$$x = r\cos\vartheta, \quad y = r\sin\vartheta, \tag{5.12}$$

reduces the system (5.11) to the form

$$\frac{dr}{dt} = rR, \quad \frac{d\vartheta}{dt} = \lambda + \Theta,$$
$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n + (p_s\cos\vartheta + q_s\sin\vartheta)\,r + X_s \tag{5.13}$$
$$(s = 1, \ldots, n),$$

where in X_s, x and y are replaced as indicated, and R and Θ denote holomorphic functions of the quantities r and x_s, vanishing at $r = x_1 = \ldots = x_n = 0$, in which the coefficients are finite sums of sines and cosines of integer multiples of ϑ with coefficients ω-periodic with respect to t. The X_s will be the same. Considering r and x_s as functions of the independent variables ϑ and t, we obtain the system

$$\frac{\partial r}{\partial t} + (\lambda + \Theta)\frac{\partial r}{\partial \vartheta} = rR,$$
$$\frac{\partial x_s}{\partial t} + (\lambda + \Theta)\frac{\partial x_s}{\partial \vartheta} = p_{s1}x_1 + \ldots + p_{sn}x_n + (p_s\cos\vartheta + q_s\sin\vartheta)\,r + X_s \tag{5.14}$$
$$(s = 1, \ldots, n).$$

For this system, Lyaponov poses the question of the existence of a formal solution of the form

$$r = c + u^{(2)}c^2 + u^{(3)}c^3 + \ldots,$$
$$x_s = u_s^{(1)}c + u_s^{(2)}c^2 + \ldots \quad (s = 1, \ldots, n), \tag{5.15}$$

where $u^{(l)}$ and $u_s^{(l)}$ are finite sums of sines and cosines of ϑ with coefficients ω-periodic as functions of t. If after a finite number of steps

Figure 12: Figure 12

it turns out that there is no such solution and the first non-periodic function $u^{(1)}$ has the form

$$u^{(1)} = gt + v(t, \vartheta), \tag{5.15$_1$}$$

where g is a non-zero constant, and v is a finite series of sines and cosines of integer multiples of ϑ with coefficients periodic with respect to t. Then the question of the stability of the zero solution of the system (5.11) Lyapunov solves by the second method. But the question becomes difficult if series of the form (5.15) can formally satisfy the system (5.14) so that all $u^{(1)}$ (t, $u_s^{(1)}$ (t, ϑ) turn out to be periodic as functions of t (or if the first non-periodic function does not have the indicated form, which is possible only when, the number $\frac{\lambda\omega}{\pi}$ is commensurable).

In all previous cases, we considered $u^{(1)}$ and $u_s^{(1)}$ only as functions of t. And if upon formal satisfaction of the given system they turned out to be periodic, then the series (5.15) also converged for small $|c|$. Now, however, they mnn to be periodic functions of t (and with respect to ϑ they are self-evidently periodic), but we cannot indicate such a $c_0 > 0$, that for $|c| < c_0$ they will converge. The question of the convergence of the series (5.15), if they turned out to be periodic [1]) functions of t, remains difficult to this day. For all points of his theory, Lyapunov usually provides examples. Here he did not indicate an example of such a system [2]) (5.11), where the series (5.15) turned out to be periodic functions of t and would converge out to be fer periodic functions of t and would converge uniformly with respect to ϑ and t for $|c| < c_0$. But, following Poincaré, Lyapunov provides an example where formally the series (5.15) are obtained as periodic functions of t. It is precisely such a case that we encounter when considering a canonical system. Let, for example,

$$\frac{dx}{dt} = -\lambda y - \frac{\partial F}{\partial y}, \quad \frac{dy}{dt} = \lambda x + \frac{\partial F}{\partial x}, \tag{5.16}$$

where $F(x, y, t+\omega) = F(x, y, t)$ is a function holomorphic with respect to x and y in the vicinity of the origin and does not have terms lower than the third dimension, i.e.

$$\left.\begin{aligned} F(x, y, t) &= \sum_{m_1+m_2\geq 3} F^{(m_1,m_2)}(t)\, x^{m_1} y^{m_2}; \\ F^{(m_1,m_2)}(t+\omega) &= F^{(m_1,m_2)}(t) \end{aligned}\right\} \tag{5.17}$$

or

$$F(x, y, t) = \sum_{m=3}^{\infty} P_m(x, y, t), \quad P_m = \sum_{k+l=m} x^k y^l \varphi_{kl}(t). \tag{5.17$_1$}$$

Substituting

$$x = r\cos\vartheta, \quad y = r\sin\vartheta, \tag{5.18}$$

we will obtain for r as a function of ϑ and t the equation

$$\frac{\partial r}{\partial t} + \left(\lambda + \frac{1}{r}\frac{\partial F}{\partial r}\right)\frac{\partial r}{\partial \vartheta} = -\frac{1}{r}\frac{\partial F}{\partial \vartheta}, \tag{5.19}$$

[1]) Note also that if $\frac{\lambda\omega}{\pi}$ is a rational number, then in determining $u^{(1)}$ and $u_s^{(1)}$ a large arbitrariness appears, to which Lyapunov draws attention.

[2]) We will consider such systems in this and the 6th paragraph.

Figure 13: Figure 13

where

$$F = F(r\cos\theta,\ \ r\sin\theta,\ t) \tag{5.20}$$

and thus

$$\frac{\partial F}{\partial r} = \frac{\partial F}{\partial x}\cos\theta + \frac{\partial F}{\partial y}\sin\theta,$$

$$\frac{\partial F}{\partial \theta} = -\frac{\partial F}{\partial x} r\sin\theta + \frac{\partial F}{\partial y} r\cos\theta. \tag{5.21}$$

Satisfying this equation with a series of the form

$$r = c + u_k c^2 + u_3 c^3 + \dots, \tag{5.22}$$
$$u_k = u_i(\sin\theta, \cos\theta, t),$$

we find u_k in the form of polynomials in $\sin\theta$, $\cos\theta$, whose coefficients will be ω-periodic functions of t. Thus we always find a formal solution of the system (5.16) in the form (5.22), where θ is a solution of the equation

$$\hat{\theta} = \lambda + \sum_{m=3}^{\infty} m r^{m-2} P_m(\cos\theta, \sin\theta, t). \tag{5.23}$$

But we are unable to prove the convergence of the series (5.22) for all sufficiently small $|c|$. We shall return to the convergence of the series (5.15). And now let us assume that we have succeeded in proving the convergence of the series (5.15). Let us see what conclusions can be drawn from this. In the previous cases, when $u^{(l)}$ and $u^{(l)}$ were determined only as functions of t and turned out to be periodic, the series (5.15) provided us with a one-parameter family of periodic solutions. Now we cannot assert this, since $u^{(l)}$ and $u_s^{(l)}$ are functions of two arguments θ and t, 2π-periodic in the first argument and ω-periodic in the second argument. But, however, this will be a boanded family of solutions and we have uniformly with respect to t

$$r^2 + x_1^2 + \dots + x_s^2 = 0 \text{ npu } c \to 0. \tag{5.24}$$

True, one must be sure that $\theta = \theta(t)$ is defined for all finite t. But we have this, since, substituting the series (5.15) into (5.13), we obtain

$$\frac{d\phi}{dt} = \lambda + \Theta(\theta, t), \tag{5.25}$$

where

$$\Theta(\theta + 2\pi, t) = \Theta(\theta, t), \tag{5.26}$$

$$\Theta(\theta, f + \omega) = \Theta(t, t) \tag{5.27}$$

and

$$\left|\frac{\partial\Theta(\theta, t)}{\partial\phi}\right| \leqslant M - \text{constant}.$$

So, let the series (5.15) converge. Then the function $\theta = \theta(t)$ is defifined for all finite t. In this assumption Lyapunov, introducing new variables $z, z_1, \dots, z_n$:

$$r = z + u_1^{(1)} z^2 + u_3^{(3)} z^3 + \dots,$$
$$z_n = z_s + u_2^{(3)} z^3 + u_4^{(3)} z^3 + \dots \tag{5.28}$$

Figure 14: Figure 14

3. There exist such systems for which the formal series (5.15) exist, but do not converge. In this case, one can construct series of the form (5.15), where the coefficients will not be periodic with respect to t (they will be quasi-periodic) or will be periodic, but not with period ω, and will converge for $|c| < c_0$.

4. One can indicate systems for which the fact of the irrationality of the $\frac{\lambda\omega}{\pi}$ does not matter when pelwing the sroblem of sctaibility of the nero pellution, but there also exist turzker systems in which for patinonal $\frac{\lambda\omega}{\pi}$ the nero solution ceases to be yctoinal possible. Circumstances changes. However, if we perssider canonical systems [12]. Let us show this.

Consider the systemy

$$\dot{x} = -\lambda y + a(t)x^2 + 2\beta(t)xy - a(t)y^2 = u(x, y, t) = u(x, y, t+\omega),$$
$$\dot{y} = \lambda x - \beta(t)x^2 + 2a(t)xy + \beta(t)y^2 = v(x, y, t) = v(x, y, t+\omega). \tag{5.29}$$

Ѣere the right-hand cides ydobnestropen vonious Cauchy-Riemans [11], noэtomy, making stony $z = x + iy$, nolyчaew

$$\frac{dz}{dt} = i\lambda z + a(t)z^2, \quad a(t) = a(t) - i\beta(t), \tag{5.30}$$

otkyga

$$z = x + iy = \frac{c(a+ib)e^{i\lambda t}}{1 - c(a+ib)\int\{A(t) - iB(t)\}\,dt}, \tag{5.31}$$

$$A(t) = a(t)\cos\lambda t + \beta(t)\sin\lambda t,$$

$$B(t) = \beta(t)\cos\lambda t - a(t)\sin\lambda t,$$

rde a, b and c — proursolvely real cocteranic constaunts. Oбosnave $\omega\lambda = 2\pi q$.

Pascmotper sgecs neckoluko csyчaes.

1. Lyct q — an irrational number. Torga $\varphi(t) = \int\{A(t) - iB(t)\}\,dt$ — organineed fynktion or z — folomorphic fynktion of c in neighterhood of $c = 0$. Holyчim

$$x + iy = u(t, c) + iv(t, c), \tag{5.32}$$

rde u and v — series in nolowiterenьnix cteners of c bes csoбodnic vlenos in exodиumous for $|c| < c_0$. Samening $x = r\cos\vartheta$, $y = r\sin\vartheta$, nolyчim

$$r = (u + iv)e^{-i\vartheta} = u(t, c)\cos\vartheta + v(t, c)\sin\vartheta. \tag{5.33}$$

We nolyved a serie ting (5.28) (uur (5.22)), no sdecs $u^i(t, \vartheta)$ moryt бытъ nenperiodivcable othocutely to t and nepiodivcable, no ne with periood ω.

Fycts, hanpumep,

$$a(t) = \cos 2\pi t, \quad \beta(t) = \sin 2\pi t, \quad \omega = 1.$$

Torga

$$z = \frac{c(a+bi)(2\pi - \lambda)e^{i\lambda t}}{2\pi - \lambda - ic(a+ib)e^{-(2\pi-\lambda)it}} \tag{5.34}$$

Figure 15: Figure 15